Relativistic aberration effect on the the light reflection law and the form of reflecting surface in a moving reference frame

V.M. Red'kov, Bernhard Rothenstein, George J. Spix

Institute of Physics, National Academy of Sciences of Belarus[1]
Politehnica University of Timisoara, Physics Department, Timisoara, Romania[2]
BSEE Illinois Institute of Technology, USA [3]

SUMMARY

The influence of the relativistic motion of the reference frame on the light reflection law is investigated. The method is based on applying the relativistic aberration affect for three light signals: incident, normal and reflected rays. The form of the reflection law in the moving reference frame is substantially modified and includes an additional parameter which is the velocity vector of the reference frame. It is shown that the reflected ray, as measured by a moving observer, would not in general be in the same plane as the incident and normal rays.

A general method to transform the form of any rigid surface in 3-dimensional space with respect to the arbitrary directed relative motion of the reference frame is detailed. This method is based on the light signals processes and the invariance of the light velocity under Lorentz transformations. It is shown that a moving observer will measure a plane surface as a hyperboloid. That observer will also measure a spherical surface as an ellipsoid. A right line in the plane is seen by a moving observer as a hyperbola.

The whole analysis is extended to the case of a uniform media.

Key words: the light reflection law, Lorentz transformation, aberration, moving mirror, rigid body, special relativity.

[1]redkov@dragon.bas-net.by
[2]bernhard rothenstein@yahoo.com
[3]gjspix@msn.com

CONTENT

1 Introduction and overview

The main idea of the present work is to follow the role of the relativistic aberration effect on the light reflection law[4]. This will result in answering the question: What is the form of the light reflection law to be used by a moving observer? The answer is of interest beyond the academic since it has always been assumed that the light reflection law is the same irrespective of the notion of the reflecting mirror. But this is not necessarily true.

The problem is old [1] and has many aspects (we plan to review them elsewhere [1-97]). In recent years in the literature have appeared a number of works in this matter. For instance, the paper by A. Gjurchinovski [27] is started by words:

Experiments involving moving mirrors are among the most interesting experiments encountered in physics. Michelson's apparatus for measuring the speed of light with a rotating wheel consisting of mirrored edges, an array of corner mirrors on the Moon's surface for estimating the distance between the Earth and the Moon, the Michelson and Morley interferometer for detecting the ether , and the rotating Sagnac interferometer for determining the angular velocity of the Earth are just a few experiments in which moving mirrors have had prominent roles. In most textbooks that discuss these experiments, it is implicitly assumed that the ordinary law of reflection of light is valid, that is, the angles of incidence and reflection are equal. Our goals in this paper are to show that the ordinary law of reflection does not hold when the mirror is moving at a constant velocity and to find a correct relation between the incident and the reflected angle.

In general points, our approach coincides with that used by A. Einstein a century ago [1]. Einstein considered the oblique incidence of a light ray on a perfectly reflecting mirror whose velocity was directed perpendicularly to its surface. To derive the equation of the angle of reflection Einstein transformed with the help of Lorentz formulas the equations describing the reflection in the reference frame where the mirror was at rest.

The specificity of this particular case is that the normal direction is the same for observers at rest and at motion. Therefore, the problem of deriving the mathematical form of the reflection law in the moving reference frame reduces to relativistic aberration effect for two light rays, incident and reflected.

In a general case this cannot be so: the surface normal can be oriented arbitrarily with respect to the direction of the motion of the reference frame. Besides, the problem must be stated in a 3-dimensional space and for any surface.

In contrast to the author of [27] – *... the angle ϕ (the inclination angle of the flat mirror in the rest reference frame) is the real physical quantity, which, by itself, has nothing to do with relativity ...* – we have accepted an opposite view. Indeed, if we consider only the rotations of the rest reference frame then we must rotate correspondingly the vector $\mathbf{n}$ of the surface normal. In the same manner, this vector $\mathbf{n}$ of the surface normal must be changed with respect to relativistic motion as well. In the mathematics of relativistic physics there do not exist any quantities that behave as 3-vectors under Euclidian rotation that will also be scalar under Lorentz transformations.

For a logical starting point we have chosen to define a light ray as a physical representative for the normal to the surface in the rest reference frame. This method from the very beginning contains a logical deficiency: in the rest reference frame one can take any of two light signals: the normal vector $\mathbf{n}$ related with the light going normally from the surface, and the inverse

[4] The same aberration considerations of the reflection law apply to the law of light refraction.

vector $\mathbf{N} = -\mathbf{n}$ associated with the light signal going to the surface. We have examined both of these variants.

In that formulating, the problem of the light reflection law in the moving reference frame is reduced to examining the details of the relativistical aberration effect for a triple of light velocity vectors. All analysis of the present paper is based only on the kinematics of relativity, that is on Lorentz transformations and the exclusive properties of the the light velocity under them. We have sought to give a complete treatment of the problem so that the reader may see many details that may have been regarded as evident and well-known.

Below we give a short overview of the content by Sections 2-19.

In Section 2, the process of reflecting the light signal on an inclined mirror in the rest reference frame K', for the simplest plan problem

$$y' = b + kx' \tag{1}$$

is stated in terms of a set of relativistic events. The normal to the inclined (flat) mirror is modeled by a light signal going normally from the mirror. In Section 3 with the help of Lorentz formulas the coordinates of the above set of events in a moving reference frame K are found.

In Section 4, the light reflection law in this simplest flat arrangement, is given in the moving reference frame

$$\frac{\sin\alpha_i + V\ (\sin\alpha - \sin\phi_2)}{(1 + V\ \cos\phi_2)} = \frac{\sin\alpha_r + V\ (\sin\alpha - \sin\phi_3)}{(1 + V\ \cos\phi_3)}\ . \tag{2}$$

It is shown that the modified form of the law, though being rather complicated, proves its invariance under the Lorentz transformations.

In Section 5, we turn to describing the geometrical form of the inclined mirror in the moving reference frame K. With the inclined flat mirror in the rest reference frame K', for the moving observer K can be associated a set of space-time events, each of those is an arrival of a light signal emitted from the space-time point $(0; 0, 0)$ toward the mirror with different angles. Evidently, all such events will take place on the surface of the mirror but at different times. These time variable can be excluded and a new equation $\varphi(x, y) = 0$ is derived, which should be considered as an equation describing the geometrical form of the mirror for the moving observer K. It should be emphasized that a definite and practically realizable procedure with the use of light signals in the reference frame K underlies this equation $\varphi(x, y) = 0$.

In Section 6 we show that the given equation $\varphi(x, y) = 0$ represents a second order curve

$$k^2\ x^2 - 2k\cosh\beta\ xy + (1 - k^2\sinh^2\beta)\ y^2 + +2bk\ \cosh\beta\ x - 2b\ y + b^2 = 0\ . \tag{3}$$

that can be translated to the canonical form by the use of a rotation and a shift. In this way, we show that the geometrical form of the mirror is a hyperbola.

In Section 7, we are to extend the results of the Section 3 to a vector form. The incident, normal, and reflected light rays in the rest reference frame K' are described by respective vectors

$$\mathbf{a}' = \frac{\mathbf{W}'_{in}}{c}\ ,\ \ \mathbf{n}' = \frac{\mathbf{W}'_{norm}}{c}\ , \qquad \mathbf{b}' = \frac{\mathbf{W}'_{out}}{c}\ .$$

all three vectors have a unit length and their lengths are invariant under the Lorentz transformations. The reflection law in the K' can be written as

$$\mathbf{a}' \times \mathbf{n}' = \mathbf{b}' \times \mathbf{n}'\ .$$

After the Lorentz transforming it will take the form

$$\frac{\mathbf{a} \times \mathbf{n} + \mathbf{V} \times (\mathbf{n} - \mathbf{a})}{1 + \mathbf{a}\mathbf{V}} = \frac{\mathbf{b} \times \mathbf{n} + \mathbf{V} \times (\mathbf{n} - \mathbf{b})}{1 + \mathbf{b}\mathbf{V}} \; . \tag{4}$$

In Section 8, some mathematical details on Lorentz transformations $L(\mathbf{V})$ are given (see much more details of the Lorentz group theory in the book [17]). With the notation $\mathbf{V} = \mathbf{e}\, th\, \beta \; , \qquad \mathbf{e}^2 = 1$, arbitrary Lorentz transformation $L(\mathbf{V})$

$$t = ch\, \beta\, t - sh\, \beta\, \mathbf{e}\, \mathbf{x} \; , \qquad \mathbf{x}' = -\mathbf{e}\, sh\, \beta\, t + \mathbf{x} + (ch\, \beta - 1)\, \mathbf{e}\, (\mathbf{e}\mathbf{x})$$

leads to the general form of the velocity addition law

$$\mathbf{w} = \frac{\mathbf{w}' - \mathbf{e}\, sh\, \beta + (ch\, \beta - 1)\, \mathbf{e}(\mathbf{e}\, \mathbf{w}')}{ch\, \beta - sh\, \beta\, \mathbf{e}\, \mathbf{w}'} \; . \tag{5}$$

In Section 9, we derive a general relationship describing the light reflection in the moving reference frame K:

$$\frac{ch\, \beta\, (\mathbf{a} \times \mathbf{n}) + sh\, \beta\, (\mathbf{a} - \mathbf{n}) \times \mathbf{e} + (ch\, \beta - 1)\, \mathbf{e}\, [\, \mathbf{e}(\mathbf{n} \times \mathbf{a})\,]}{ch\, \beta + sh\, \beta\, (\mathbf{e}\mathbf{a})} =$$
$$= \frac{ch\, \beta\, (\mathbf{b} \times \mathbf{n}) + sh\, \beta(\mathbf{b} - \mathbf{n}) \times \mathbf{e} + (ch\, \beta - 1)\, \mathbf{e}\, [\, \mathbf{e}(\mathbf{n} \times \mathbf{b})\,]}{ch\, \beta + sh\, \beta\, (\mathbf{e}\mathbf{b})} \; . \tag{6}$$

Take notice of terms including $[\, \mathbf{e}(\mathbf{n} \times \mathbf{a})\,]$ which substantially extend the previous result (4) of the plane problem.

In the Section 9, a special attention is given to one other aspect of the problem. In the rest reference frame K' three vectors $\mathbf{a}, \mathbf{n}, \mathbf{b}$ belong to the same single plane, but for a moving observer it is not so. To describe this phenomenon, we have introduced the following quantity (the volumes of the light parallelepiped $[\, \mathbf{n}\, (\mathbf{a} \times \mathbf{b})\,]$):

$$\Delta' = \mathbf{n}'\, (\mathbf{a}' \times \mathbf{b}') = 0 \qquad \text{but} \qquad \Delta = \mathbf{n}\, (\mathbf{a} \times \mathbf{b}) =$$
$$= \frac{-2\, sh\, \beta\, (\mathbf{a}'\mathbf{n}')\, [\, \mathbf{e}(\mathbf{n}' \times \mathbf{a}')\,]}{(ch\, \beta - sh\, \beta\, \mathbf{e}\mathbf{n}')\, (ch\, \beta - sh\, \beta\, \mathbf{e}\mathbf{a}')\, (ch\, \beta - sh\, \beta\, \mathbf{e}\mathbf{b}')} \; . \tag{7}$$

In Section 10 we turn to examining the form of the reflection surface, the plane in the rest reference frame ($\mathbf{d}$ is a normal to the plane)

$$S' : \qquad \mathbf{x}'\, \mathbf{d}' + D = 0 \; ,$$

in the moving reference frame K. For this with the reflection plane is associated the following set of space-time events:

$$S' : \qquad \Longrightarrow \qquad \Big\{ \; t' = \sqrt{\mathbf{x}'^2} \; , \; \mathbf{x}'\, \mathbf{d}' + D' = 0 \; \Big\} \; .$$

This set of events can be readily transformed to the moving reference frame, which after excluding the time-variable provides us with a new surface equation

$$S : \qquad (\mathbf{x}\mathbf{d}) + (\mathbf{e}\mathbf{d}) \; \Big[\; sh\, \beta\, (\; \sqrt{\mathbf{x}^2}\;) + (ch\, \beta - 1)\, (\mathbf{e}\mathbf{x}) \; \Big] + D = 0 \; . \tag{8}$$

Take notice than only when $\mathbf{ed} = 0$ takes place, then the plane in the rest frame will look as the plane in the moving reference frame too. For all other cases the plane does not preserve its geometrical form and is a second order surface.

In Section 11 a canonical form of this second order surface has been established with the use of a special rotation and a shift. It turns to be a hyperboloid. Its symmetry axis is directed along the vector

$$\mathbf{f} = \frac{\mathbf{d} + (ch\,\beta - 1)(\mathbf{ed})\mathbf{e}}{(\mathbf{ed})\ sh\,\beta} \ .$$

In Section 12, the geometrical form of the (plane) circle in the rest reference frame has been transformed to the moving observer, it proves to be the ellipse

$$\frac{(x - 2R\ sh\,\beta)^2}{R^2\ ch^2\beta} + \frac{y^2}{R^2} = 1 \ . \tag{9}$$

In Section 13, a 3-dimensional problem is solved: the sphere in the rest reference frame, being transformed to the moving reference frame, becomes an ellipsoid

$$\frac{(X - 2R\ sh\,\beta)^2}{R^2\ ch^2\beta} + \frac{Y^2}{R^2} + \frac{Z^2}{R^2} = 1 \ , \tag{10}$$

its symmetry axis is directed along the vector $\mathbf{V}$.

In Section 14, we turn to the case when the light ray (incident, reflected, and normal) are propagated in a uniform media with the refraction index $n > 1$. In the rest reference frame K' the reflection law preserves its form. However, there arise differences when going to a moving reference frame K. The fact of the most significance is that when using ordinary (vacuum-based) Lorentz transformations then because the speed of light in the rest media is less than c ($kc < c$) the modulus of the light velocity vector does not preserve its value.

General mathematical form of the reflection law in the moving reference frame formally stays the same, however one must take into account that the lengths of the vectors involved $\mathbf{a}, \mathbf{b}, \mathbf{n}$ are different from 1. The length of the light vector in the moving reference frame in presence of the uniform media is

$$W = \sqrt{1 - \frac{(1 - k^2)}{(ch\,\beta - (\mathbf{eW}')\ sh\,\beta)^2}} \ . \tag{11}$$

It should be especially noted one other aspect of the problem: the latter equation means that the light velocity in the reference frame K is a function of direction of the propagation of the light. This fact is of most significance because it changes basically the general structure of special relativity in presence of a media. In such circumstances there appears an absolute reference frame related to the rest media, the reference frame K'. In the reference frame K', the light velocity is an isotropic quantity that preserves its value in all space directions. In any other reference frame, moving K, the light velocity is anisotropic – it is a function of directions.

In Section 15, some aspects the tensor formalism of 4-velocities u^a are specified for the light case. In the Section 16. some geometrical aspects of the relativistic velocity concept in terms of the Lobachewsky 3-geometry are briefly discussed [....].

In Section 17, the relativistic transformation of the geometrical form $\varphi(\mathbf{x}') = 0$ of any surface to a moving reference frame in presence of a media is considered:

$$\varphi(\mathbf{x}') = 0 \qquad \Longrightarrow \qquad \varphi \left[\mathbf{x} + \mathbf{e} \; (\; sh \; \beta \; \frac{\sqrt{\mathbf{x}^2}}{\sqrt{\mathbf{W}^2}} \; + (ch \; \beta - 1) \; (\mathbf{ex}) \right] = 0 \; , \tag{12}$$

In Section 18, we briefly discuss the scheme of Special Relativity in a uniform media that can be constructed on the base of the light velocity in the media kc [....]. Modified Lorentz formulas look as

$$t' = \frac{t + \mathbf{V}\mathbf{x}/k^2c^2}{\sqrt{1 - V^2/k^2c^2}} \; , \qquad \mathbf{x}' = \mathbf{x} - \mathbf{e} \; (\mathbf{ex}) + \frac{\mathbf{e}(\mathbf{ex}) + \mathbf{V}t}{\sqrt{1 - V^2/k^2c^2}} \; .$$

The value of light velocity in the media is invariant under modified Lorentz formula: $\mathbf{W}^2 = k^2c^2 \;\Longrightarrow\; \mathbf{W}'^2 = k^2c^2$.

In Section 19, the corresponding scheme of transforming the form of a surface in going to the moving reference frame is discussed. The general method is the same:

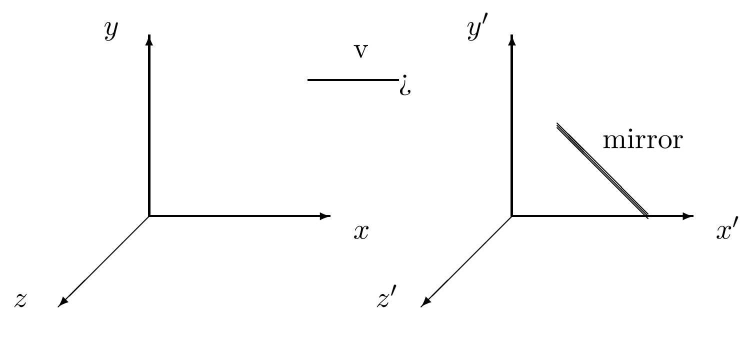
$$S: \qquad \varphi \; [\; \mathbf{x} + \mathbf{e} \; (\; sh \; \beta \; \sqrt{\mathbf{x}^2} + (ch \; \beta - 1) \; \mathbf{ex} \;) \;] = 0 \; ,$$

the media's presence enters through the hyperbolic functions

$$ch \; \beta = \frac{1}{\sqrt{1 - V^2/k^2c^2}} \; , \qquad sh \; \beta = \frac{V}{kc \; \sqrt{1 - V^2/k^2c^2}} \; .$$

2 Problem setting

Let two observers (two inertial reference frames), K and K', be given. The standard arrangement is shown in the Fig. 1



reference frame K reference frame K'

Fig. 1 Problem setting

At the moment of its coincidence, event 1 ($t_0 = 0, x_0 = 0$) and event 1′ ($t_0 = 0, x'_0 = 0$), observer K' sends a light ray signal which is reflected by an inclined flat mirror, unmoving in reference frame K', at the moment t'_2 (event 2′); the third event 3′ is taken by symmetry considerations – see in Fig. 2

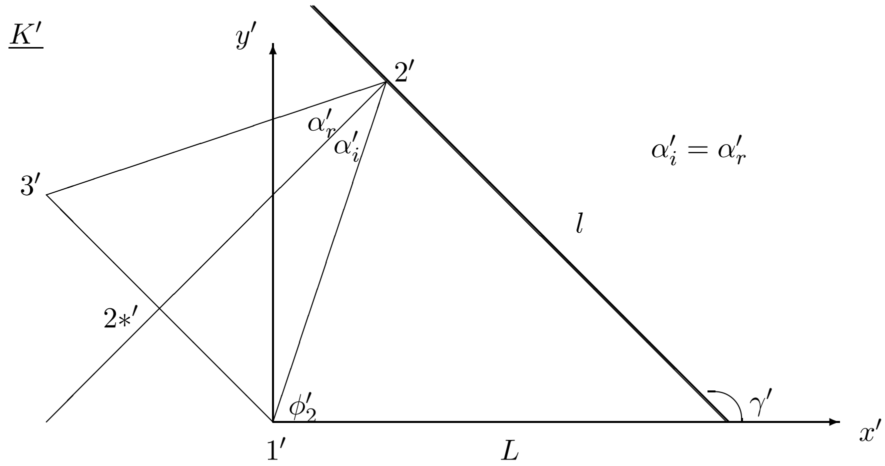


Fig. 2 Reflection and relativistic events

How will this phenomenon be seen for second observer K?

From geometry considerations in Fig. 2 it follows (take notice in Fig.2 the angles $\gamma', \phi'_2, \phi'_3$ are measured from zero degree position at the axis x; γ' and Φ'_3 are obtuse, whereas ϕ'_2 is an acute angle)

$$(180^0 - \gamma') + \phi'_2 + (90^0 - \alpha'_i) = 180^0 : \qquad \Longrightarrow \qquad (13)$$
$$\alpha'_i = \phi'_2 - \gamma' + 90^0 \; .$$

In K' frame the motion of the light along the line $1'2'$ is given by parametric formulas

$$x'(t') = (c\ \cos\phi'_2)\ t' \;, \qquad y'(t') = (c\ \sin\phi'_2)\ t' \;;$$

and the inclined mirror's form is given by

$$x' = L' + l'\ \cos\gamma' \;, \qquad y' = l'\ \sin\gamma' \;, \qquad (14)$$

At the point where the light falls on the mirror <u>(event $2'$)</u> it must hold two relations:

$$(c\ \cos\phi'_2)\ t'_2 = L' + l'_2\ \cos\gamma' \;,$$
$$(c\ \sin\phi'_2)\ t'_2 = l'_2\ \sin\gamma' \; .$$

This linear system under t'_2, l'_2

$$t'_2\ (c\ \cos\phi'_2) - l'_2\ \cos\gamma' = L' \;,$$
$$t'_2(\ c\ \sin\phi'_2) - l'_2\ \sin\gamma' = 0 \; .$$

can be easily solved

$$t'_2 = \frac{L}{c}\ \frac{\sin\gamma'}{(\cos\phi'_2\ \sin\gamma' - \sin\phi'_2\ \cos\gamma')} = \frac{L}{c}\ \frac{\sin\gamma'}{\sin(\gamma' - \phi'_2)} \;,$$
$$l'_2 = L\ \frac{\sin\phi'_2}{(\cos\phi'_2 \sin\gamma' - \sin\phi'_2 \cos\gamma')} = L\ \frac{\sin\phi'_2}{\sin(\gamma' - \phi'_2)} \; .$$

Thus, to <u>the event $2'$</u> there corresponds a set of coordinates:

$$t'_2 = \frac{L}{c}\ \frac{\sin\gamma'}{\sin(\gamma' - \phi'_2)} \;,$$
$$x'_2 = (c\ \cos\phi'_2)\ t'_2 = \frac{L\ \cos\phi'_2\ \sin\gamma'}{\sin(\gamma' - \phi'_2)} \;,$$
$$y'_2 = (c\ \sin\phi'_2)\ t'_2 = \frac{L\ \sin\phi'_2\ \sin\gamma'}{\sin(\gamma' - \phi'_2)} \;; \qquad (15)$$

at an established position of the mirror (L, γ') in K' frame, a given angle ϕ'_2 determines unambiguously coordinates of the event $2'$: x'_2, y'_2, t'_2.

Let us write an equation for light trajectory $2'3'$ after reflection at the point $2'$. From geometry considerations in Fig 2. it follows an expression for its directing angle ϕ'_3:

$$\phi'_3 = (\phi'_2 - \alpha'_i + 180^0 - \alpha'_i) = \phi'_2 + 180^0 - 2(\phi'_2 - \gamma' + 90^0) : \qquad \Longrightarrow$$
$$\phi'_3 = 2\,\gamma' - \phi'_2 \tag{16}$$

therefore the light at the way $2'3'$ is described by

$$\begin{aligned} x'(t') - x'_2 &= (t' - t'_2)\ c\ \cos\ \phi'_3\ , \\ y'(t') - y'_2 &= (t' - t'_2)\ c\ \sin\ \phi'_3\ . \end{aligned} \tag{17}$$

Let us use established convention to determine event $3'$ in a symmetrical way with respect to event $1'$: $t'_3 = 2t'_2$. As a result coordinates of the event $3'$ are

$$\begin{aligned} t'_3 &= 2\ \frac{L}{c}\ \frac{\sin\gamma'}{\sin(\gamma' - \phi'_2)}\ , \\ x'_3 &= x'_2 + t'_2\ c\ \cos\ \phi'_3\ , \\ y'_3 &= y'_2 + t'_2\ c\ \sin\ \phi'_3\ . \end{aligned} \tag{18}$$

It is to be noted that both sets $(x'_2, y'_2, t'_2, \phi'_3)$ and $(x'_3;, y'_3, t'_3)$ are definite and uniquely determined functions of the initial angular parameter ϕ'_2.

One other point: to have possibility to describe a normal to the mirror (with a directing angle $\alpha' = \gamma' + 90^0$) in terms of events of space-time world, we should introduce one subsidiary event $2*'$. Let at the reflection point observer K' send an additional light signal along normal to the mirror

$$\begin{aligned} x'^*_2 &= x'_2 + \cos\alpha'\ (t'^*_2 - t'_2)\ , \\ y'^*_2 &= y'_2 + \sin\alpha'\ (t'^*_2 - t'_2)\ , \\ t'^*_2 &= 2t'_2\ , \qquad \alpha' = \gamma' + 90^0\ . \end{aligned} \tag{2.9}$$

NOTATION

In the following we are to translate coordinates of four events $1' - 2' - 2*' - 3'$ to the reference frame K, with the help of the Lorentz formulas

$$t = \frac{t' - Vx'/c^2}{\sqrt{1 - V^2/c^2}}\ , \qquad x = \frac{x' - Vt'}{\sqrt{1 - V^2/c^2}}\ , \qquad y = y'\ . \tag{19}$$

and then we are to analyze relationships between four events $1 - 2 - 2* - 3$ in the reference frame K.

We will now employ the convention of letting $c = 1$ to simplify the formulas (sometimes it is referred as the use of unit system with $c = 1$):

$$ct \Longrightarrow t, \qquad \frac{V}{c} \Longrightarrow V\ \ ,$$

then Lorentz transforms (19) take the form

$$t = \frac{t' - V x'}{\sqrt{1 - V^2}} \; , \qquad x = \frac{x' - V t'}{\sqrt{1 - V^2}} \; , \qquad y = y' \; . \tag{20}$$

With this notation the formulas we need are

<u>Events $2'$</u>, $3'$, $2*'$:

$$\begin{aligned} t'_2 &= L \, \frac{\sin \gamma'}{\sin(\gamma' - \phi'_2)} \; , \\ x'_2 &= \cos \phi'_2 \, t'_2 = \frac{L \, \cos \phi'_2 \, \sin \gamma'}{\sin(\gamma' - \phi'_2)} \; , \\ y'_2 &= \sin \phi'_2 \, t'_2 = \frac{L \, \sin \phi'_2 \, \sin \gamma'}{\sin(\gamma' - \phi'_2)} \; . \end{aligned}$$

$$\begin{aligned} t'_3 &= 2 \, L \, \frac{\sin \gamma'}{\sin(\gamma' - \phi'_2)} \; , \\ x'_3 &= x'_2 + t'_2 \, \cos \, \phi'_3 \; , \\ y'_3 &= y'_2 + t'_2 \, \sin \, \phi'_3 \; , \end{aligned}$$

$$t'_{2*} = t'_2 \; , \qquad \begin{aligned} x'_{2*} &= x'_2 - \, t'_2 \, \cos \alpha'_i \; , \\ y'_{2*} &= y'_2 - \, t'_2 \, \sin \alpha'_i \; . \end{aligned} \tag{21}$$

3 Transition from the fixed frame K' to the moving K

With the use of (20), for coordinates of the event 2 one easily gets

$$\begin{aligned} x_2 &= \frac{x'_2 - V \, t'_2}{\sqrt{1 - V^2}} = \frac{\cos \phi'_2 - V}{\sqrt{1 - V^2}} \, t'_2 \; , \\ t_2 &= \frac{t'_2 - V \, x'_2}{\sqrt{1 - V^2}} = \frac{1 - V \, \cos \phi'_2}{\sqrt{1 - V^2}} \, t'_2 \; , \\ y_2 &= y'_2 = \sin \phi_2 \, t'_2 \; . \end{aligned} \tag{22}$$

Expressing from second equation t'_2 as a function of t_2:

$$t'_2 = \frac{\sqrt{1 - V^2}}{1 - V \, \cos \phi'_2} \, t_2 \; ,$$

for coordinates (x, y) of the event 2 we have

$$\begin{aligned} x_2 &= \frac{\cos \phi'_2 - V}{1 - V \, \cos \phi'_2} \; t_2 \equiv \cos \phi_2 \, t_2 \; , \\ y_2 &= \sin \phi'_2 \; \frac{\sqrt{1 - V^2}}{1 - V \, \cos \phi'_2} \; t_2 \equiv \sin \phi_2 \, t_2 \; . \end{aligned} \tag{23}$$

It is readily shown that the introduction into (23) of a variable angle ϕ_2 in the reference frame K is correct. Verifying is reduced to

$$(\cos\phi_2)^2 + (\sin\phi_2)^2 = (\frac{\cos\phi_2' - V}{1 - V\ \cos\phi_2'})^2 + (\sin\phi_2'\ \ \frac{\sqrt{1-V^2}}{1 - V\ \cos\phi_2'})^2 =$$

$$= \frac{\cos^2\phi_2' - 2V\cos\phi_2' + V^2 + \sin^2\phi_2' - \sin^2\phi_2' V^2}{(1 - V\ \cos\phi_2')^2} = \frac{1 - 2V\cos\phi_2' + \cos^2\phi_2' V^2}{(1 - V\ \cos\phi_2')^2} = 1\ .$$

In essence, here is tested the famous Einstein's postulate on light velocity constancy: its invariance under Lorentz transformations.

Let us write down the formulas for changing the direction of light velocity vector at transfer from one inertial reference frame to another (in other words this phenomenon is called the light aberration):

$$\cos\phi_2 = \frac{\cos\phi_2' - V}{1 - V\ \cos\phi_2'}\ , \qquad \sin\phi_2 = \sin\phi_2'\ \ \frac{\sqrt{1-V^2}}{1 - V\ \cos\phi_2'}\ . \tag{24}$$

Their inverse is naturally symmetrical

$$\cos\phi_2' = \frac{\cos\phi_2 + V}{1 + V\ \cos\phi_2}\ , \qquad \sin\phi_2' = \sin\phi_2\ \ \frac{\sqrt{1-V^2}}{1 + V\ \cos\phi_2}\ .$$

On the same way we carry out converting the coordinates of the event $3'$:

$$\left.\begin{aligned} t_3' &= 2t_2'\ , \\ x_3' &= x_2' + (t_3' - t_2')\ \cos\ \phi_3'\ , \\ y_3' &= y_2' + (t_3' - t_2')\ \sin\ \phi_3'\ . \end{aligned}\right\} \qquad \Longrightarrow$$

$$t_3 = \frac{t_3' - V\ x_3'}{\sqrt{1-V^2}} = \frac{2t_2' - V\ (\ \cos\phi_2'\ t_2' + t_2'\ \cos\ \phi_3'\ )}{\sqrt{1-V^2}} =$$

$$= \frac{2 - V\ (\ \cos\phi_2' + \cos\ \phi_3'\ )}{\sqrt{1-V^2}}\ t_2'\ ,$$

from where with the use of

$$t_2 = \frac{1 - V\cos\phi_2'}{\sqrt{1-V^2}}\ \ t_2'$$

it follows

$$t_3 - t_2 = \frac{1 - V\ \cos\ \phi_3'}{\sqrt{1-V^2}}\ \ t_2'\ . \tag{3.6c}$$

Analogously, from

$$x_3 = \frac{x_3' - V\ t_3'}{\sqrt{1-V^2}} = \frac{\cos\phi_2'\ t_2' + t_2'\ \cos\ \phi_3' - V\ 2t_2'}{\sqrt{1-V^2}} =$$

$$= \frac{\cos\phi_2' + \cos\ \phi_3' - 2V}{\sqrt{1-V^2}}\ t_2'\ ,$$

with the use of

$$x_2 = \frac{\cos\phi'_2 - V}{\sqrt{1-V^2}}\; t'_2 \;,$$

one finds

$$x_3 - x_2 = \frac{\cos\;\phi'_3 - V}{\sqrt{1-V^2}}\; t'_2 \;. \tag{25}$$

And finally,

$$y_3 = y'_3 = y'_2 + (t'_3 - t'_2)\sin\phi'_3 = y_2 + \;t'_2\;\sin\phi'_3 \;: \qquad \Longrightarrow$$
$$y_3 - y_2 = t'_2\;\;\sin\phi'_3 \;. \tag{26}$$

Thus, the trajectory 23 in the K frame is given by

$$\begin{aligned} t_3 - t_2 &= \frac{1 - V\;\cos\;\phi'_3}{\sqrt{1-V^2}}\; t'_2 \;, \\ x_3 - x_2 &= \frac{\cos\;\phi'_3 - V}{1 - V\cos\phi'_3}\;(t_3 - t_2) \equiv \cos\phi_3\;(t_3 - t_2)\;, \\ y_3 - y_2 &= \;\sin\phi'_3\;\frac{\sqrt{1-V^2}}{1-V\cos\phi'_3}\;(t_3 - t_2) \equiv \sin\phi_3\;(t_3 - t_2)\;. \end{aligned} \tag{27}$$

In other words, the light aberration at the way $2-3$ is described by relations

$$\frac{\cos\;\phi'_3 - V}{1 - V\cos\phi'_3} = \cos\phi_3\;, \qquad \sin\phi'_3\;\frac{\sqrt{1-V^2}}{1-V\cos\phi'_3} = \sin\phi_3\;, \tag{28}$$

the inverse transform looks as

$$\frac{\cos\;\phi_3 + V}{1 + V\cos\phi_3} = \cos\phi'_3\;, \qquad \sin\phi_3\;\frac{\sqrt{1-V^2}}{1+V\cos\phi_3} = \sin\phi'_3\;.$$

In the same way, we readily produce the formulas describing the trajectory $2-2*$ (see. (3.9)):

$$\begin{aligned} t^*_2 - t_2 &= \frac{1 - V\;\cos\;\alpha'}{\sqrt{1-V^2}}\; t'_2 \;, \\ x^*_2 - x_2 &= \frac{\cos\;\alpha' - V}{1 - V\cos\alpha'}\;(t^*_2 - t_2) \equiv \cos\alpha\;(t^*_2 - t_2)\;, \\ y^*_2 - y_2 &= \;\sin\alpha'\;\frac{\sqrt{1-V^2}}{1-V\cos\alpha'}\;(t^*_2 - t_2) \equiv \sin\alpha\;(t^*_2 - t_2)\;. \end{aligned} \tag{29}$$

The light aberration on the way $2-2*$ is given by

$$\frac{\cos\;\alpha' - V}{1 - V\cos\alpha'} = \cos\alpha\;, \qquad \sin\alpha'\;\frac{\sqrt{1-V^2}}{1-V\cos\alpha'} = \sin\alpha\;, \tag{30}$$

and inverse ones

$$\frac{\cos\;\alpha + V}{1 + V\cos\alpha} = \cos\alpha'\;, \qquad \sin\alpha\;\frac{\sqrt{1-V^2}}{1+V\cos\alpha} = \sin\alpha'\;.$$

4 On relativistic form of the light reflection law

In the reference frame K' the light reflection law is formulated in the equality

$$\alpha_i' = \alpha_r' \; , \qquad\qquad (31)$$
$$\alpha_i' = \phi_2' - (\alpha' - 180^0) \; , \qquad \alpha_r' = \alpha' - \phi_3' \; .$$

With the use of the Lorentz formulas one can transform eq.(31) to the moving reference frame. Thereby it will be obtained a generalized form of the light reflection on the moving mirror.

Generally, one might expect one of the two following possibilities:

1) either equation $\alpha_i' = \alpha_r'$ turns out to be invariant under the Lorentz transformation

$$\alpha_i' = \alpha_r' \qquad \Longrightarrow \qquad \alpha_i = \alpha_r \; ,$$
$$\alpha_i = 180^0 - (\alpha - \phi_2) \; , \qquad \alpha_r = (\alpha - \phi_3) \; . \qquad\qquad (32)$$

2) or after Lorentz transformation eqs. (31) will take a modified form

$$\alpha_i' = \alpha_r' \qquad \Longrightarrow \qquad \varphi(\alpha_i, \alpha_r; V) = 0 \; ,$$
$$\alpha_i = 180^0 - (\alpha - \phi_2) \; , \qquad \alpha_r = (\alpha - \phi_3) \; , \qquad\qquad (33)$$

and one should expect that this new equation $\varphi(\alpha_i, \alpha_r; V) = 0$ is Lorentz invariant;, in other words it is covariant under Lorentz transformation.

Now we have to carry out the calculation needed. Firstly, it will be better to re-write eqs. (31) in the form

$$\cos\alpha_i' = \cos\alpha_r' \qquad \Longrightarrow \qquad -\cos(\alpha' - \phi_2') = \cos(\alpha' - \phi_3') \; ,$$
$$\sin\alpha_i' = \sin\alpha_r' \qquad \Longrightarrow \qquad \sin(\alpha' - \phi_2') = \sin(\alpha' - \phi_3') \; . \qquad\qquad (34)$$

Now, these equation are to be transformed to an unprimed quantities with the use of the formulas

$$\cos\alpha' = \frac{\cos\alpha + V}{1 + V\ \cos\alpha} \; , \qquad \sin\alpha' = \sin\alpha\ \frac{\sqrt{1-V^2}}{1 + V\ \cos\alpha} \; ,$$
$$\cos\phi_2' = \frac{\cos\phi_2 + V}{1 + V\ \cos\phi_2} \; , \qquad \sin\phi_2' = \sin\phi_2\ \frac{\sqrt{1-V^2}}{1 + V\ \cos\phi_2} \; ,$$
$$\cos\phi_3' = \frac{\cos\phi_3 + V}{1 + V\ \cos\phi_3} \; , \qquad \sin\phi_3' = \sin\phi_3\ \frac{\sqrt{1-V^2}}{1 + V\ \cos\phi_3} \; . \qquad\qquad (35)$$

The left-hand part of the first equation in (34) becomes

$$-\cos(\alpha' - \phi_2') = -(\cos\alpha'\ \cos\phi_2' + \sin\alpha'\ \sin\phi_2') =$$
$$-\left[\frac{\cos\alpha + V}{(1 + V\ \cos\alpha)}\ \frac{\cos\phi_2 + V}{(1 + V\ \cos\phi_2)} + \sin\alpha\ \frac{\sqrt{1-V^2}}{(1 + V\ \cos\alpha)}\ \sin\phi_2\ \frac{\sqrt{1-V^2}}{(1 + V\ \cos\phi_2)} \right] =$$
$$= -\ \frac{\cos(\alpha - \phi_2) + V\ (\cos\alpha + \cos\phi_2) + V^2(1 - \sin\alpha\ \sin\phi_2)}{(1 + V\ \cos\alpha)\ (1 + V\ \cos\phi_2)} \; , \qquad\qquad (36)$$

and the right-hand part will look

$$\cos(\alpha' - \phi_3') = \frac{\cos(\alpha - \phi_3) + V\ (\cos\alpha + \cos\phi_3) + V^2(1 - \sin\alpha\ \sin\phi_3)}{(1 + V\ \cos\alpha)\ (1 + V\ \cos\phi_3)}\ . \tag{37}$$

Thus, the first equation in (34) after Lorentz transformation is as follows

$$\begin{aligned} &\frac{-\cos(\alpha - \phi_2) - V\ (\cos\alpha + \cos\phi_2) - V^2(1 - \sin\alpha\ \sin\phi_2)}{(1 + V\ \cos\phi_2)} = \\ &= \frac{\cos(\alpha - \phi_3) + V\ (\cos\alpha + \cos\phi_3) + V^2(1 - \sin\alpha\ \sin\phi_3)}{(1 + V\ \cos\phi_3)}\ , \end{aligned} \tag{38}$$

or

$$\begin{aligned} &\frac{\cos\alpha_i - V\ (\cos\alpha + \cos\phi_2) - V^2(1 - \sin\alpha\ \sin\phi_2)}{(1 + V\ \cos\phi_2)} = \\ &= \frac{\cos\alpha_r + V\ (\cos\alpha + \cos\phi_3) + V^2(1 - \sin\alpha\ \sin\phi_3)}{(1 + V\ \cos\phi_3)}\ . \end{aligned} \tag{39}$$

Analogously, consider other equation in (34). The left-hand and right-hands parts become respectively

$$\begin{aligned} &\sin(\alpha' - \phi_2') = \sin\alpha'\ \cos\phi_2' - \cos\alpha'\ \sin\phi_2' = \\ &= \sin\alpha\ \frac{\sqrt{1 - V^2}}{(1 + V\ \cos\alpha)}\ \frac{\cos\phi_2 + V}{(1 + V\ \cos\phi_2)} - \frac{\cos\alpha + V}{(1 + V\ \cos\alpha)}\ \sin\phi_2\ \frac{\sqrt{1 - V^2}}{(1 + V\ \cos\phi_2)} = \\ &= \frac{\sqrt{1 - V^2}}{(1 + V\ \cos\alpha)\ (1 + V\ \cos\phi_2)}\ [\ \sin(\alpha - \phi_2) + V\ (\sin\alpha - \sin\phi_2)\ ]\ , \end{aligned} \tag{40}$$

and

$$\sin(\alpha' - \phi_3') = \frac{\sqrt{1 - V^2}}{(1 + V\ \cos\alpha)\ (1 + V\ \cos\phi_3)}\ [\ \sin(\alpha - \phi_3) + V\ (\sin\alpha - \sin\phi_3)\ ]\ . \tag{41}$$

So that equation in (34) after Lorentz transformation looks as

$$\frac{\sin(\alpha - \phi_2) + V\ (\sin\alpha - \sin\phi_2)}{(1 + V\ \cos\phi_2)} = \frac{\sin(\alpha - \phi_3) + V\ (\sin\alpha - \sin\phi_3)}{(1 + V\ \cos\phi_3)}\ , \tag{42}$$

or

$$\frac{\sin\alpha_i + V\ (\sin\alpha - \sin\phi_2)}{(1 + V\ \cos\phi_2)} = \frac{\sin\alpha_r + V\ (\sin\alpha - \sin\phi_3)}{(1 + V\ \cos\phi_3)}\ . \tag{43}$$

Thus, the light reflection law in the reference frame K':

$$\begin{aligned} \cos\alpha_i' = \cos\alpha_r' \qquad &\Longrightarrow \qquad -\cos(\alpha' - \phi_2') = \cos(\alpha' - \phi_3')\ , \\ \sin\alpha_i' = \sin\alpha_r' \qquad &\Longrightarrow \qquad \sin(\alpha' - \phi_2') = \sin(\alpha' - \phi_3')\ , \end{aligned}$$

after translating to the moving reference frame K' take on the form

$$
\frac{-\cos(\alpha-\phi_2) - V\ (\cos\alpha + \cos\phi_2) - V^2(1-\sin\alpha\ \sin\phi_2)}{(1+V\ \cos\phi_2)} =
$$

$$
= \frac{\cos(\alpha-\phi_3) + V\ (\cos\alpha + \cos\phi_3) + V^2(1-\sin\alpha\ \sin\phi_3)}{(1+V\ \cos\phi_3)} \ , \tag{44}
$$

$$
\frac{\sin(\alpha-\phi_2) + V\ (\sin\alpha - \sin\phi_2)}{(1+V\ \cos\phi_2)} = \frac{\sin(\alpha-\phi_3) + V\ (\sin\alpha - \sin\phi_3)}{(1+V\ \cos\phi_3)} \ . \tag{45}
$$

The relations obtained seem rather cumbersome. Nevertheless they should be taken seriously because they display the property of Lorentz-invariance. Such an additional test consists in the following: let the frame K, in turn, be moving with the velocity $\tilde{V}$ with respect to another reference frame $\tilde{K}$. We should expect invariance of equations (44)-(45) under corresponding Lorentz transformations:

$$
\begin{array}{ll}
\cos\alpha = \dfrac{\cos\tilde{\alpha} + \tilde{V}}{1+\tilde{V}\ \cos\tilde{\alpha}} \ , & \sin\alpha = \sin\tilde{\alpha}\ \dfrac{\sqrt{1-\tilde{V}^2}}{1+\tilde{V}\ \cos\tilde{\alpha}} \ , \\
\cos\phi_2 = \dfrac{\cos\tilde{\phi}_2 + \tilde{V}}{1+\tilde{V}\ \cos\tilde{\phi}_2} \ , & \sin\phi_2 = \sin\tilde{\phi}_2\ \dfrac{\sqrt{1-\tilde{V}^2}}{1+\tilde{V}\ \cos\tilde{\phi}_2} \ , \\
\cos\phi_3 = \dfrac{\cos\tilde{\phi}_3 + \tilde{V}}{1+\tilde{V}\ \cos\tilde{\phi}_3} \ , & \sin\phi_3 = \sin\tilde{\phi}_3\ \dfrac{\sqrt{1-\tilde{V}^2}}{1+\tilde{V}\ \cos\tilde{\phi}_3} \ .
\end{array} \tag{46}
$$

Before proceeding to calculation, one useful change in variables used should be done: instead of the light velocity $\tilde{V}$ (and all other ones) it is better to employ a hyperbolic variable:

$$
\frac{1}{\sqrt{1-\tilde{V}^2}} = \cosh\tilde{\beta}, \qquad \frac{\tilde{V}}{\sqrt{1-\tilde{V}^2}} = \sinh\tilde{\beta}, \qquad \tilde{V} = \tanh\tilde{\beta}\ .
$$

Eqs.(46) take the form (the same applies to (35))

$$
\begin{array}{ll}
\cos\alpha = \dfrac{\cosh\tilde{\beta}\ \cos\tilde{\alpha} + \sinh\tilde{\beta}}{\cosh\tilde{\nu} + \sinh\tilde{\beta}\ \cos\tilde{\alpha}} \ , & \sin\alpha = \dfrac{\sin\tilde{\alpha}}{\cosh\tilde{\nu} + \sinh\tilde{\beta}\ \cos\tilde{\alpha}} \ , \\
\cos\phi_2 = \dfrac{\cosh\tilde{\beta}\ \cos\tilde{\phi}_2 + \sinh\tilde{\beta}}{\cosh\tilde{\nu} + \sinh\tilde{\beta}\ \cos\tilde{\phi}_2} \ , & \sin\phi_2 = \dfrac{\sin\tilde{\phi}_2}{\cosh\tilde{\nu} + \sinh\tilde{\beta}\ \cos\tilde{\phi}_2} \ , \\
\cos\phi_3 = \dfrac{\cosh\tilde{\beta}\ \cos\tilde{\phi}_3 + \sinh\tilde{\beta}}{\cosh\tilde{\nu} + \sinh\tilde{\beta}\ \cos\tilde{\phi}_3} \ , & \sin\phi_2 = \dfrac{\sin\tilde{\phi}_3}{\cosh\tilde{\nu} + \sinh\tilde{\beta}\ \cos\tilde{\phi}_3} \ .
\end{array} \tag{47}
$$

Firstly let us consider behavior of the more simple formula (45) under Lorentz transformation (47). For the first factor we have

$$
\frac{1}{1+V\ \cos\phi_2} = \frac{1}{1+\tanh\beta\ (\cosh\tilde{\beta}\ \cos\tilde{\phi}_2 + \sinh\tilde{\beta})\ /\ (\cosh\tilde{\nu} + \sinh\tilde{\beta}\ \cos\tilde{\phi}_2)} =
$$

$$
\frac{1+\tanh\tilde{\beta}\ \cos\tilde{\phi}_2}{(\tanh\tilde{\beta} + \tanh\beta)\cos\tilde{\phi}_2 + (1+\tanh\beta\ \tanh\tilde{\beta})} \ ;
$$

from where with the use of

$$1+\tanh\beta\ \tanh\tilde\beta=\frac{\tanh\tilde\beta+\tanh\beta}{\tanh(\beta+\tilde\beta)}\ ,$$

get to

$$\begin{aligned}\frac{1}{1+V\ \cos\phi_2}&=\frac{1+\tanh\tilde\beta\ \cos\tilde\phi_2}{1+\tanh\beta\ \tanh\tilde\beta}\ \frac{1}{1+\tanh(\beta+\tilde\beta)\ \cos\tilde\phi_2}=\\&=\frac{\cosh\beta\ (\cosh\tilde\beta+\sinh\tilde\beta\ \cos\tilde\phi_2)}{\sinh\beta\ \sinh\tilde\beta+\cosh\beta\ \cosh\tilde\beta}\ \frac{1}{1+\tanh(\beta+\tilde\beta)\ \cos\tilde\phi_2}\ .\end{aligned}\tag{48}$$

Now consider the term

$$\begin{aligned}&\sin(\alpha-\phi_2)=\sin\alpha\ \cos\phi_2-\cos\alpha\ \sin\phi_2=\\&=\frac{\sin\tilde\alpha}{(\cosh\tilde\beta+\sinh\tilde\beta\ \cos\tilde\alpha)}\ \frac{\cosh\tilde\beta\ \cos\tilde\phi_2+\sinh\tilde\beta}{(\cosh\tilde\beta+\sinh\tilde\beta\ \cos\tilde\phi_2)}\ -\\&-\ \frac{\cosh\tilde\beta\ \cos\tilde\alpha+\sinh\tilde\beta}{(\cosh\tilde\beta+\sinh\tilde\beta\ \cos\tilde\alpha)}\ \frac{\sin\tilde\phi_2}{(\cosh\tilde\beta+\sinh\tilde\beta\ \cos\tilde\phi_2)}=\\&=\frac{\cosh\tilde\beta\ \sin(\tilde\alpha-\tilde\phi_2)+\sinh\tilde\beta\ \ (\sin\tilde\alpha-\sin\tilde\phi_2)}{(\cosh\tilde\beta+\sinh\tilde\beta\ \cos\tilde\alpha)\ (\cosh\tilde\beta+\sinh\tilde\beta\ \cos\tilde\phi_2)}\ .\end{aligned}\tag{49}$$

And finally, the term

$$\begin{aligned}&V\ (\sin\alpha-\sin\phi_2)=\\&=\frac{\sinh\beta}{\cosh\beta}\ \left[\frac{\sin\tilde\alpha}{\cosh\tilde\nu+\sinh\tilde\beta\ \cos\tilde\alpha}-\frac{\sin\tilde\phi_2}{\cosh\tilde\nu+\sinh\tilde\beta\ \cos\tilde\phi_2}\right]=\\&=\frac{\sinh\beta}{\cosh\beta}\ \frac{\sinh\tilde\beta\ \sin(\tilde\alpha-\tilde\phi_2)+\cosh\tilde\beta\ (\sin\tilde\alpha-\sin\tilde\phi_2)}{(\cosh\tilde\beta+\sinh\tilde\beta\ \cos\tilde\alpha)\ (\cosh\tilde\beta+\sinh\tilde\beta\ \cos\tilde\phi_2)}\ .\end{aligned}\tag{50}$$

Combining (49) and (50), we have (see. (45))

$$\begin{aligned}&\sin(\alpha-\phi_2)+V(\sin\alpha-\sin\phi_2)=\frac{1}{(\cosh\tilde\beta+\sinh\tilde\beta\cos\tilde\alpha)(\cosh\tilde\beta+\sinh\tilde\beta\cos\tilde\phi_2)}\times\\&\times\left[\sin(\alpha-\phi_2)\left(\cosh\tilde\beta+\frac{\sinh\beta}{\cosh\beta}\sinh\tilde\beta\right)+(\sin\tilde\alpha-\sin\tilde\phi_2)\left(\sinh\tilde\beta+\frac{\sinh\beta}{\cosh\beta}\cosh\tilde\beta\right)\right]=\\&=\frac{1}{(\cosh\tilde\beta+\sinh\tilde\beta\cos\tilde\alpha)(\cosh\tilde\beta+\sinh\tilde\beta\cos\tilde\phi_2)}\times\\&\times\left[\sin(\alpha-\phi_2)(\frac{\cosh\beta\cosh\tilde\beta+\sinh\beta\sinh\tilde\beta}{\cosh\beta}+(\sin\tilde\alpha-\sin\tilde\phi_2)\ \ \frac{\cosh\beta\sinh\tilde\beta+\sinh\beta\cosh\tilde\beta}{\cosh\beta}\right]\ .\end{aligned}$$

Now we arrive at

$$\begin{aligned}&\frac{\sin(\alpha-\phi_2)+V\ (\sin\alpha-\sin\phi_2)}{(1+V\ \cos\phi_2)}=\frac{1}{1+\tanh(\beta+\tilde\beta)\ \cos\tilde\phi_2}\times\\&\times\left[\sin(\tilde\alpha-\tilde\phi_2)+\ (\sin\tilde\alpha-\sin\tilde\phi_2)\ \ \frac{\cosh\beta\ \sinh\tilde\beta+\sinh\beta\ \cosh\tilde\beta}{\sinh\beta\ \sinh\tilde\beta+\cosh\beta\ \cosh\tilde\beta}\right]\ .\end{aligned}$$

From this, using the hyperbolic function identity

$$\tanh(\beta + \tilde{\beta}) = \frac{\sinh(\beta + \tilde{\beta})}{\cosh(\beta + \tilde{\beta})} = \frac{\cosh\beta\ \sinh\tilde{\beta} + \sinh\beta\ \cosh\tilde{\beta}}{\sinh\beta\ \sinh\tilde{\beta} + \cosh\beta\ \cosh\tilde{\beta}} ,$$

for the left-hand part of (45) we arrive at the following

$$\frac{\sin(\alpha - \phi_2) + \tanh\beta\ (\sin\alpha - \sin\phi_2)}{(1 + \tanh\beta\ \cos\phi_2)} =$$
$$= \frac{\sin(\tilde{\alpha} - \tilde{\phi}_2) +\ \tanh(\beta + \tilde{\beta})\ \ (\sin\tilde{\alpha} - \sin\tilde{\phi}_2)}{1 + \tanh(\beta + \tilde{\beta})\ \cos\tilde{\phi}_2} \ . \tag{51}$$

For the right-hand part of (45) , with no additional calculation we will obtain

$$\frac{\sin(\alpha - \phi_3) + \tanh\beta\ (\sin\alpha - \sin\phi_3)}{(1 + \tanh\beta\ \cos\phi_3)} =$$
$$= \frac{\sin(\tilde{\alpha} - \tilde{\phi}_3) +\ \tanh(\beta + \tilde{\beta})\ \ (\sin\tilde{\alpha} - \sin\tilde{\phi}_3)}{1 + \tanh(\beta + \tilde{\beta})\ \cos\tilde{\phi}_3} \ . \tag{52}$$

Therefore, equation (45) after the Lorentz transformation to the reference frame $\tilde{K}$ displays a required invariant form

$$\frac{\sin(\tilde{\alpha} - \tilde{\phi}_2) +\ \tanh(\beta + \tilde{\beta})\ \ (\sin\tilde{\alpha} - \sin\tilde{\phi}_2)}{1 + \tanh(\beta + \tilde{\beta})\ \cos\tilde{\phi}_2} =$$
$$= \frac{\sin(\tilde{\alpha} - \tilde{\phi}_3) +\ \tanh(\beta + \tilde{\beta})\ \ (\sin\tilde{\alpha} - \sin\tilde{\phi}_3)}{1 + \tanh(\beta + \tilde{\beta})\ \cos\tilde{\phi}_3} \ . \tag{53}$$

Remember the relativistic velocity addition rule in terms of hyperbolic variables:

$$\frac{V + \tilde{V}}{1 + V\ \tilde{V}} \qquad \Longrightarrow \qquad \tanh(\beta + \tilde{\beta}) = \frac{\tanh\beta + \tanh\tilde{\beta}}{1 + \tanh\beta\ \tanh\tilde{\beta}} \ .$$

CONCLUSION

The light reflection law on moving mirror (reference frame K)

$$\frac{\sin(\alpha - \phi_2) + V\ (\sin\alpha - \sin\phi_2)}{(1 + V\ \cos\phi_2)} = \frac{\sin(\alpha - \phi_3) + V\ (\sin\alpha - \sin\phi_3)}{(1 + V\ \cos\phi_3)} \ ,$$

or

$$\frac{\sin\alpha_i + V\ (\sin\alpha - \sin\phi_2)}{(1 + V\ \cos\phi_2)} = \frac{\sin\alpha_r + V\ (\sin\alpha - \sin\phi_3)}{(1 + V\ \cos\phi_3)} \ ,$$

is invariant under Lorentz transformations. We will not verify the relativistic invariance of the second condition (44); evidently it is the case.

5 On describing the form of a moving mirror

Section concerns a general problem, described in [31] as follows:

As is well known, the notion of a rigid body, which proves so useful in Newtonian mechanics, is incompatible with the existence of a universal finite upper bound for all signal velocities [Laue - 1911]. As a result, the notion of a perfectly rigid body does not exist within the framework of SR. However, the notion of a rigid motion does exist. Intuitively speaking, a body moves rigidly if, locally, the relative spatial distances of its material constituents are unchanging .

Let us turn to the geometrical form of the mirror in unmoving reference frame K'; it is described by two parametric equations

$$x' = L' + l' \ \cos\gamma' \ , \qquad y' = l' \ \sin\gamma' \ . \tag{54}$$

After evident manipulation we get to

$$\frac{x'-L'}{\cos\gamma'} = l' \ : \qquad \Longrightarrow \qquad y' = \frac{x'-L'}{\cos\gamma'} \ \sin\gamma' \ ,$$
$$y' = -L \ \tan \ \gamma' + \tan \ \gamma' \ x' \ ,$$

or

$$y' = b + k \ x' \tag{55}$$

where $b = -L \ \tan \ \gamma'$ and $k = \tan \ \gamma'$are fixed parameters defining the straight line – contour of the mirror.

For another observer K, to the mirror there corresponds a set of space-time events of the type $2'$, each of those is an arrival of a light signal emitted from the space-time point $1' : -(0; 0, 0)$ toward the mirror with different angles ϕ'_2. Evidently, all such events will take place on the surface of the mirror but at different times:

$$t'_2 = L \ \frac{\sin\gamma'}{\sin(\gamma' - \phi'_2)} \ ,$$
$$x'_2 = \cos\phi'_2 \ \ t'_2 \ , \qquad y'_2 = \sin\phi'_2 \ \ t'_2 \ . \tag{56}$$

Space-time coordinates of these events in the reference fame K may be found through the Lorentz formulas:

$$t'_2 = \frac{t_2 + V x_2}{\sqrt{1-V^2}} = \frac{x_2/\cos\phi_2 + V x_2}{\sqrt{1-V^2}} = \frac{1/\cos\phi_2 + V}{\sqrt{1-V^2}} \ x_2 \ ,$$
$$x'_2 = \frac{x_2 + V t_2}{\sqrt{1-V^2}} = \frac{x_2 + V x_2/\cos\phi_2}{\sqrt{1-V^2}} = \frac{1 + V/\cos\phi_2}{\sqrt{1-V^2}} \ x_2 \ ,$$
$$y'_2 = y_2 \ .$$

Taking them into (55) we will arrive at the equation

$$y_2 = b + k \ \frac{1 + V/\cos\phi_2}{\sqrt{1-V^2}} \ \ x_2 \ . \tag{57}$$

With the help of

$$x_2 = \cos\phi_2 \; t_2 \; , \qquad y_2 = \sin\phi_2 \; t_2 : \qquad \tan\phi_2 = \frac{y_2}{x_2} \; , \qquad \Longrightarrow$$

$$\frac{1}{\cos\phi_2} = \sqrt{1+\tan^2\phi_2} = \sqrt{1+\frac{y_2^2}{x_2^2}}$$

eq. (57) may be rewritten as follows:

$$y_2 = b + k \; \frac{1 + V\sqrt{1+y_2^2/x_2^2}}{\sqrt{1-V^2}} \; x_2$$

or (for simplicity the index 2 at the coordinates will be omitted)

$$y = b + k \; (\; \cosh\beta + \sinh\beta \; \sqrt{1+\frac{y^2}{x^2}} \;) \; x \; . \tag{58}$$

Relationship (58) should be considered as an equation describing the surface of the moving mirror, it is not evidently a straight line equation. It should be emphasized again that a definite and practically realizable procedure with the use of light signals in the reference frame K underlies this equation.

6 On geometrical form of a moving mirror

What curve is given by the equation (58):

$$y = b + k \; [\; \cosh\beta + \sinh\beta \; \sqrt{1+\frac{y^2}{x^2}} \;] \; x \; .$$

After simple rewriting from (58) it follows

$$(y - b - k\cosh\beta \; x)^2 = k^2 \; \sinh^2\beta \; (x^2+y^2) \; ,$$

and further

$$k^2 \; x^2 - 2k\cosh\beta \; xy + (1 - k^2\sinh^2\beta) \; y^2 + \\ +2bk \; \cosh\beta \; x - 2b \; y + b^2 = 0 \; . \tag{59}$$

This is of second order curve. In order to establish its explicit geometrical form eq. (59) should be translated to the canonical form. We will proceed in accordance with standard procedures. As a first step let us find certain (x, y)-rotation that will eliminate a coefficient at the cross term xy from the curve equation. To this end, instead of (x, y) one should introduce new (rotated) variables (X', Y')

$$\begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix} \begin{vmatrix} X' \\ Y' \end{vmatrix} \; . \tag{60}$$

Then eq. (59) will take the form

$$
\begin{aligned}
k^2\ (\cos\phi\ X' - \sin\phi\ Y')^2 + (1 - k^2\sinh^2\beta)(\sin\phi\ X' + \cos\phi\ Y')^2 - \\
-2k\cosh\beta\ (\cos\phi\ X' - \sin\phi\ Y')(\sin\phi\ X' + \cos\phi\ Y') + \\
+2bk\ \cosh\beta\ (\cos\phi\ X' - \sin\phi\ Y') - \\
-2b\ (\sin\phi\ X' + \cos\phi\ Y') + b^2 = 0
\end{aligned}
$$

or

$$
\begin{aligned}
X'^2[\ k^2\cos^2\phi + (1 - k^2\sinh^2\beta)\sin^2\phi - 2k\cosh\beta\cos\phi\sin\phi\ ] + \\
+Y'^2[\ k^2\sin^2\phi + (1 - k^2\sinh^2\beta)\cos^2\phi + 2k\cosh\beta\cos\phi\sin\phi\ ]\ + \\
+X'Y'[-2\sin\phi\cos\phi k^2 + (1 - k^2\sinh^2\beta)2\sin\phi\cos\phi - 2k\cosh\beta(\cos^2\phi - \sin^2\phi)] + \\
+X'[2bk\cosh\beta\cos\phi - 2b\sin\phi] + Y'[2bk\cosh\beta\sin\phi - 2b\cos\phi] + b^2 = 0\ .
\end{aligned}
\tag{61}
$$

The rotation angle $\phi = \phi_0$ should be determined from special requirement that the coefficient at $X'Y'$ be zero:

$$
\sin 2\phi_0\ [\ (1 - k^2\sinh^2\beta)\ -\ k^2\ ] - 2k\ \cosh\beta\ \cos 2\phi_0 = 0\ ,
$$

from where it follows

$$
\tan 2\phi_0 = \frac{2k\ \cosh\beta}{1 - k^2\ \cosh^2\beta}\ . \tag{62}
$$

With the aid of the known relation

$$
\tan 2\phi_0 = \frac{2\ \tan\phi_0}{1 - \tan^2\phi_0}
$$

we arrive at a simple representation for $\tan\phi_0$:

$$
\tan\phi_0 = k\ \cosh\beta\ , \qquad \text{or} \quad \tan\phi_0 = \tan\gamma'\ \cosh\beta\ \ , \tag{63}
$$

and also expressions for

$$
\begin{aligned}
\sin\phi_0 = \frac{\tan\phi_0}{\sqrt{1 + \tan^2\phi_0}} = \frac{k\cosh\beta}{\sqrt{1 + k^2\cosh^2\beta}}\ , \\
\cos\phi_0 = \frac{1}{\sqrt{1 + \tan^2\phi_0}} = \frac{1}{\sqrt{1 + k^2\cosh^2\beta}}\ .
\end{aligned}
$$

Now, eq. (61) becomes

$$
\begin{aligned}
X'^2\ \frac{k^2 + (1 - k^2\sinh^2\beta)\ k^2\cosh^2\beta - 2k\cosh\beta\ k\cosh\beta}{1 + k^2\cosh^2\beta} + \\
+Y'^2\ \frac{k^2\ k^2\cosh^2\beta + (1 - k^2\sinh^2\beta) + 2k\cosh\beta\ k\cosh\beta}{1 + k^2\cosh^2\beta} + \\
+\frac{X'}{\sqrt{1 + k^2\cosh^2\beta}}\ (2bk\ \cosh\beta - 2bk\ \cosh\beta) + \\
+\frac{Y'}{\sqrt{1 + k^2\cosh^2\beta}}\ (-2bk\ \cosh\beta\ k\cosh\beta - 2b)\ + b^2 = 0\ .
\end{aligned}
\tag{64}
$$

From this, after simple computation we get to the equation of second order:

$$
\begin{aligned}
&- X'^2 \, k^2 \sinh^2 \beta + Y'^2 \, (1 + k^2) - \\
&-2b \sqrt{1 + k^2 \cosh^2 \beta} \, Y' + b^2 = 0 \, ;
\end{aligned}
\tag{65}
$$

which is an equation of a hyperbola. Its canonical form will be achieved by a definite displacement along the axis Y':

$$
- X'^2 k^2 \sinh^2 \beta + (1 + k^2)[Y' - b\frac{\sqrt{1 + k^2 \cosh^2 \beta}}{1 + k^2}\,]^2 - b^2 \frac{1 + k^2 \cosh^2 \beta}{1 + k^2} + b^2 = 0 \, ;
$$

that is

$$
- X'^2 k^2 \sinh^2 \beta + (1 + k^2)[Y' - b\frac{\sqrt{1 + k^2 \cosh^2 \beta}}{1 + k^2}\,]^2 = \frac{b^2 k^2 \sinh^2 \beta}{1 + k^2} \, . \tag{66}
$$

So, the canonical equation for a geometrical form of the moving mirror in the frame K looks

$$
\begin{gathered}
- \frac{X'^2}{A^2} + \frac{(Y' - C)^2}{B^2} = +1 \, , \\
A^2 = \frac{b^2}{1 + k^2} \, , \qquad B^2 = \frac{b^2 k^2 \sinh^2 \beta}{(1 + k^2)^2} \, , \qquad C = \frac{b \, \sqrt{1 + k^2 \cosh^2 \beta}}{(1 + k^2)} \, .
\end{gathered}
\tag{67}
$$

Let us trace again establishing canonical equation: it consists of two steps: rotation and displacement:

$$
\begin{aligned}
x = \cos \phi_0 \, X' - \sin \phi_0 \, Y' \, , \qquad & y = \sin \phi_0 \, X' + \cos \phi_0 \, Y' \, ; \\
& X' = X'' \, , \qquad Y'' = Y' - C \, .
\end{aligned}
$$

After performing these two transforms the equation of the moving mirror will represent the hyperbola:

$$
- \frac{X''^2}{A^2} + \frac{Y''^2}{B^2} = +1 \, ;
$$

The above rotation (6.7a) may be clarified by Figure 3:

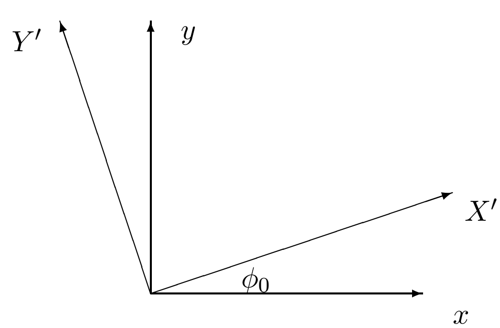


Fig. 3 Additional rotation

To the total transform there corresponds to Figure 4

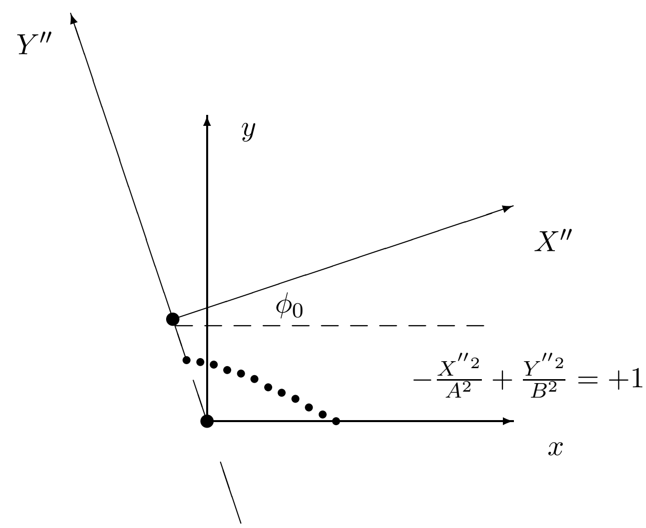


Fig. 4 The mirror's form is a hyperbola in (X'', Y'') frame

OVERALL RESULT:

1) the moving mirror for the observer K looks as

$$k^2\, x^2 - 2k \cosh\beta\; xy + (1 - k^2 \sinh^2\beta)\; y^2 + \\ +2bk\; \cosh\beta\; x - 2b\; y + b^2 = 0\; ;$$

2) in the reference frame K one needs to take the new coordinates X'', Y''

$$\begin{aligned} x &= \cos\phi_0\; X'' - \sin\phi_0\; (Y'' + C) = \\ &= -\sin\phi_0\; C + (\cos\phi_0\; X'' - \sin\phi_0\; Y'')\; , \\ y &= \sin\phi_0\; X'' + \cos\phi_0\; (Y'' + Y'_0) = \\ &= +\cos\phi_0\; C + (\sin\phi_0\; X'' + \cos\phi_0\; Y'')\; , \end{aligned}$$

$$\tan\phi_0 = k\; \cosh\beta\; , \qquad C = \frac{b\,\sqrt{1 + k^2\cosh^2\beta}}{(1 + k^2)}\; ;$$

3) then eq. (6.8a) will become a canonical form of a hyperbola

$$-\frac{X'^2}{A^2} + \frac{Y''^2}{B^2} = +1\; ; \qquad A^2 = \frac{b^2}{1 + k^2}\; , \qquad B^2 = \frac{b^2 k^2 \sinh^2\beta}{(1 + k^2)^2}\; .$$

7 Generalization and simplification, vector form

Now we are to extend the results above to a more general vector form. Let us start with the conventional designation. The incident and reflected light rays in the rest reference frame K'

are described by respective vectors of unit length $\mathbf{a}'$ and $\mathbf{b}'$:

$$\mathbf{a}' = \frac{\mathbf{W}'_{in}}{c} \ , \ \ \mathbf{a}'^2 = 1 \ , \qquad \mathbf{b}' = \frac{\mathbf{W}'_{out}}{c} \ , \ \ \mathbf{b}'^2 = 1 \ ; \tag{68}$$

With the surface perpendicular there can be associated the unit vector $\mathbf{n}'$ (normal light ray – see Fig. 5)

$$\mathbf{n}' = \frac{\mathbf{W}'_{norm}}{c} \ , \qquad \mathbf{n}'^2 = 1 \ . \tag{69}$$

The designation introduced can be clarified by Fig. 5

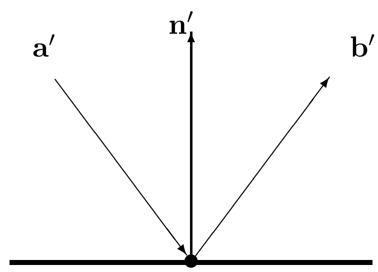


Fig 5. Reflection in the rest reference frame K'

The reflection law in the K' can be mathematically described by means of the following vector formula (all three vectors have a unit length)

$$\mathbf{a}' \times \mathbf{n}' = \mathbf{b}' \times \mathbf{n}' \ . \tag{70}$$

This relationship includes both

1) $\alpha'_i = \alpha'_r = \alpha'$ and
2) all three vectors belong to the same plane.

In addition the following relation is in effect:

$$\cos\alpha'_i = \cos\alpha'_r \qquad \Longrightarrow \qquad \mathbf{a}'\,\mathbf{n}' + \mathbf{b}'\,\mathbf{n}' = 0 \ . \tag{71}$$

To describe the same reflection process in the moving reference frame K means that the relation (70) is to be transformed to this new frame through the Lorentz formula. The Lorentz formulas we need are

$$\begin{aligned}
a'_1 &= \frac{a_1 + V}{1 + a_1 V} \ , \qquad a'_2 = \frac{\sqrt{1-V^2}}{1 + a_1 V}\, a_2 \ , \qquad a'_3 = \frac{\sqrt{1-V^2}}{1 + a_1 V}\, a_3 \ , \\
b'_1 &= \frac{b_1 + V}{1 + b_1 V} \ , \qquad b'_2 = \frac{\sqrt{1-V^2}}{1 + b_1 V}\, b_2 \ , \qquad b'_3 = \frac{\sqrt{1-V^2}}{1 + b_1 V}\, b_3 \ , \\
n'_1 &= \frac{n_1 + V}{1 + n_1 V} \ , \qquad n'_2 = \frac{\sqrt{1-V^2}}{1 + n_1 V}\, n_2 \ , \qquad n'_3 = \frac{\sqrt{1-V^2}}{1 + n_1 V}\, n_3 \ .
\end{aligned} \tag{72}$$

The most significant property of all the three light velocity vectors consists in the following: their length is invariant under the Lorentz transformation[5]. Indeed, for example let it be $(n_1'^2 +$

[5]This statement is true only if all light rays are propagated in the vacuum.

$n_2'^2 + n_3'^2) = 1$, then

$$n_1^2 + n_2^2 + n_3^2 = \frac{(n_1' - V)^2}{(1 - n_1'V)^2}) + \frac{1 - V^2}{(1 - n_1 V)^2}\, n_2'^2 + \frac{1 - V^2}{(1 - n_1 V)^2}\, n_3'^2$$
$$= \frac{1}{(1 - n_1 V)^2} \quad \left[(n_1'^2 + n_2'^2 + n_3'^2) - 2n_1' V + V^2\, (1 - n_2'^2 - n_2'^2 \right] = 1\;.$$

The vector relation (70) in coordinate form looks as three ones

$$\begin{array}{rl} 1: & a_2' n_3' - a_3' n_2' = b_2' n_3' - b_3' n_2'\;, \\ 2: & a_3' n_1' - a_1' n_3' = b_3' n_1' - b_1' n_3'\;, \\ 3: & a_1' n_2' - a_2' n_1' = b_1' n_2' - b_2' n_1'\;. \end{array}$$

From these, substituting eqs. (72), we get to

$$\begin{array}{rl} 1: & \dfrac{a_2 n_3 - a_3 n_2}{1 + a_1 V} = \dfrac{b_2 n_3 - b_3 n_2}{1 + b_1 V}\;, \\ 2: & \dfrac{a_3(n_1 + V) - (a_1 + V) n_3}{1 + a_1 V} = \dfrac{b_3(n_1 + V) - (b_1 + V) n_3}{1 + b_1 V}\;, \\ 3: & \dfrac{(a_1 + V) n_2 - a_2 (n_1 + V)}{1 + a_1 V} = \dfrac{(b_1 + V) n_2 - b_2 (n_1 + V)}{1 + b_1 V}\;, \end{array}$$

or differently

$$\begin{array}{rl} 1: & \dfrac{a_2 n_3 - a_3 n_2}{1 + a_1 V} = \dfrac{b_2 n_3 - b_3 n_2}{1 + b_1 V}\;, \\ 2: & \dfrac{(a_3 n_1 - a_1 n_3) - V(n_3 - a_3)}{1 + a_1 V} = \dfrac{(b_3 n_1 - b_1 n_3) - V(n_3 - b_3)}{1 + b_1 V}\;, \\ 3: & \dfrac{(a_1 n_2 - a_2 n_1) + V(n_2 - a_2)}{1 + a_1 V} = \dfrac{(b_1 n_2 - b_2 n_1) + V(n_2 - b_2)}{1 + b_1 V}\;. \end{array} \qquad (73)$$

Now, having in mind

$$\begin{array}{r} \mathbf{V} = (V, 0, 0)\;, \\ \mathbf{V} \times (\,\mathbf{n} - \mathbf{a}) = (0, -V(n_3 - a_3), V(n_2 - a_2)\,)\;, \\ \mathbf{V} \times (\mathbf{n} - \mathbf{b}) = (\,0, -V(n_3 - b_3), V(n_2 - b_2)\,)\;, \end{array}$$

the three equations (73) can be written as a vector one

$$\frac{\mathbf{a} \times \mathbf{n} + \mathbf{V} \times (\mathbf{n} - \mathbf{a})}{1 + \mathbf{aV}} = \frac{\mathbf{b} \times \mathbf{n} + \mathbf{V} \times (\mathbf{n} - \mathbf{b})}{1 + \mathbf{bV}}\;. \qquad (74)$$

Because

$$\mid \mathbf{a} \times \mathbf{n} \mid = \sin \alpha_i, \qquad \mid \mathbf{b} \times \mathbf{n} \mid = \sin \alpha_r\;,$$

the formula (74) represents the reflection law in the moving reference frame K.

Let us compare the generalized formula (74) with previously examined a particular case when

$$\begin{aligned} a_1 &= \cos\phi_2 \ , & a_2 &= \sin\phi_2 \ , & a_3 &= 0 \ , \\ b_1 &= \cos\phi_3 \ , & b_2 &= \sin\phi_3 \ , & b_3 &= 0 \ , \\ n_1 &= \cos\alpha \ , & n_2 &= \sin\alpha \ , & n_3 &= 0 \ , \end{aligned} \tag{75}$$

and eqs. (73) will take the form

$$\begin{gathered} 0 = 0 \ , \qquad\qquad 0 = 0 \ , \\ \frac{(\cos\phi_2 \sin\alpha - \sin\phi_2 \cos\alpha) + V(\sin\alpha - \sin\phi_2)}{1 + \cos\phi_2 V} = \\ = \frac{(\cos\phi_3 \sin\alpha - \sin\phi_3 \cos\alpha) + V(\sin\alpha - \sin\phi_3)}{1 + \cos\phi_3 V} \ , \end{gathered}$$

or

$$\frac{\sin(\alpha - \phi_2) + V(\sin\alpha - \sin\phi_2)}{1 + \cos\phi_2 V} = \frac{\sin(\alpha - \phi_3) + V(\sin\alpha - \sin\phi_3)}{1 + \cos\phi_3 V} \ , \tag{76}$$

which coincides with (42).

Now let us transform to the reference frame K the second relevant relationship (71):

$$\begin{gathered} \mathbf{a}' \, \mathbf{n}' + \mathbf{b}' \, \mathbf{n}' = 0 \ , \qquad \Longrightarrow \\ \frac{a_1 + V}{1 + a_1 V} \, \frac{n_1 + V}{1 + n_1 V} + \frac{(1 - V^2)(a_2 n_2 + a_3 n_3)}{(1 + a_1 V)(1 + n_1 V)} \ + \\ + \frac{b_1 + V}{1 + b_1 V} \, \frac{n_1 + V}{1 + n_1 V} + \frac{(1 - V^2)(b_2 n_2 + b_3 n_3)}{(1 + b_1 V)(1 + n_1 V)} = 0 \ , \end{gathered}$$

from this it follows

$$\begin{gathered} \frac{(a_1 n_1 + a_2 n_2 + a_3 n_3) + (a_1 + n_1)V + V^2(1 - a_2 n_2 - a_3 n_3)}{1 + a_1 V} + \\ + \frac{(b_1 n_1 + b_2 n_2 + b_3 n_3) + (b_1 + n_1)V + V^2(1 - b_2 n_2 - b_3 n_3)}{1 + b_1 V} = 0 \ . \end{gathered} \tag{77}$$

For the case (75) it becomes

$$\begin{gathered} \frac{(\cos\phi_2 \cos\alpha + \sin\phi_2 \sin\alpha) + (\cos\phi_2 + \cos\alpha)V + V^2(1 - \sin\phi_2 \sin\alpha)}{1 + \cos\phi_2 V} + \\ + \frac{(\cos\phi_3 \cos\alpha + \sin\phi_3 \sin\alpha) + (\cos\phi_3 + \cos\alpha)V + V^2(1 - \sin\phi_3 \sin\alpha)}{1 + \cos\phi_3 V} = 0 \end{gathered}$$

or

$$\begin{gathered} \frac{\cos(\alpha - \phi_2) + (\cos\phi_2 + \cos\alpha)V + V^2(1 - \sin\phi_2 \sin\alpha)}{1 + \cos\phi_2 V} + \\ + \frac{\cos(\alpha - \phi_3) + (\cos\phi_3 + \cos\alpha)V + V^2(1 - \sin\phi_3 \sin\alpha)}{1 + \cos\phi_3 V} = 0 \end{gathered} \tag{78}$$

which coincides with eq. (44):

$$\frac{-\cos(\alpha-\phi_2)-V\ (\cos\alpha+\cos\phi_2)-V^2(1-\sin\alpha\ \sin\phi_2)}{(1+V\ \cos\phi_2)}=$$
$$=\frac{\cos(\alpha-\phi_3)+V\ (\cos\alpha+\cos\phi_3)+V^2(1-\sin\alpha\ \sin\phi_3)}{(1+V\ \cos\phi_3)}=0\ .$$

8 The Lorentz transform with the arbitrary velocity vector V

We shall now investigate the case of an arbitrary velocity vector $\mathbf{V}$. The velocity equations of (73) and rewritten as one equation in (74) are true only when $\mathbf{V}=(V,0,0)$.

Firstly, in the following we will need an explicit form of the Lorentz transform with an arbitrary $\mathbf{V}$. With the notation

$$\begin{gathered}\mathbf{V}=\mathbf{e}\,th\ \beta\ ,\qquad \mathbf{e}^2=1\ ,\\ \frac{1}{\sqrt{1-V^2}}=ch\ \beta,\qquad \frac{V}{\sqrt{1-V^2}}=sh\ \beta,\end{gathered}\tag{79}$$

the Lorentz matrix is [86]

$$(L_a{}^b)=$$
$$\begin{vmatrix} ch\ \beta & e_1 sh\ \beta & e_2 sh\ \beta e_2 & e_3 sh\ \beta \\ e_1 sh\ \beta & ch\ \beta-(ch\ \beta-1)(e_2^2+e_3^2) & (ch\ \beta-1)e_1e_2 & (ch\ \beta-1)e_2e_3 \\ e_2 sh\ \beta & (ch\ \beta-1)e_1e_2 & ch\ \beta-(ch\ \beta-1)(e_1^2+e_3^2) & (ch\ \beta-1)e_2e_3 \\ e_3 sh\ \beta & (ch\ \beta-1)e_1e_3 & (ch\ \beta-1)e_2e_3 & ch\ \beta-(ch\ \beta-1)(e_1^2+e_2^2) \end{vmatrix}.$$

In three cases the matrix L looks especially simple:

$$\mathbf{e}=(1,0,0),\qquad L=\begin{vmatrix} ch\ \beta & sh\ \beta & 0 & 0\\ sh\ \beta & ch\ \beta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix},$$
$$\mathbf{e}=(0,1,0),\qquad L=\begin{vmatrix} ch\ \beta & 0 & sh\ \beta & 0\\ 0 & 1 & 0 & 0\\ sh\ \beta & 0 & ch\ \beta & 0\\ 0 & 0 & 0 & 1\end{vmatrix},$$
$$\mathbf{e}=(0,0,1),\qquad L=\begin{vmatrix} ch\ \beta & 0 & 0 & sh\ \beta\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ sh\ \beta & 0 & 0 & ch\ \beta\end{vmatrix}.$$

The matrix L can be rewritten as follows

$$(L_a{}^b)=\begin{vmatrix} ch\ \beta & e_1 sh\ \beta & e_2 sh\ \beta e_2 & e_3 sh\ \beta \\ e_1 sh\ \beta & 1+(ch\ \beta-1)e_1^2 & (ch\ \beta-1)e_1e_2 & (ch\ \beta-1)e_2e_3 \\ e_2 sh\ \beta & (ch\ \beta-1)e_1e_2 & 1+(ch\ \beta-1)e_2^2 & (ch\ \beta-1)e_2e_3 \\ e_3 sh\ \beta & (ch\ \beta-1)e_1e_3 & (ch\ \beta-1)e_2e_3 & 1+(ch\ \beta-1)e_3^2 \end{vmatrix}$$

or in a more formal symbolical manner

$$L = \left| \begin{array}{cc} ch\,\beta & \mathbf{e}\,sh\,\beta \\ \mathbf{e}\,sh\,\beta & [\delta_{ij} + (ch\,\beta - 1) e_i e_j] \end{array} \right| . \qquad (80)$$

The Lorentz transform (80) acts on space-time coordinates $(t, \mathbf{x})$ in accordance with

$$L: \qquad t' = ch\,\beta\,t + sh\,\beta\,\mathbf{e}\,\mathbf{x}\,,$$
$$\mathbf{x}' = \mathbf{e}\,sh\,\beta\,t + \mathbf{x} + (ch\,\beta - 1)\,\mathbf{e}\,(\mathbf{ex})\,. \qquad (81)$$

For the inverse transform we have

$$L^{-1}: \qquad t = ch\,\beta\,t' - sh\,\beta\,\mathbf{e}\,\mathbf{x}'\,,$$
$$\mathbf{x} = -\mathbf{e}\,sh\,\beta\,t + \mathbf{x}' + (ch\,\beta - 1)\,\mathbf{e}\,(\mathbf{ex}')\,.$$

Indeed, let us consider the time variable t:

$$\begin{aligned} t = ch\,\beta\,t' - sh\,\beta\,\mathbf{e}\,\mathbf{x}' = \\ = ch\,\beta\,(ch\,\beta\,t + sh\,\beta\,\mathbf{e}\,\mathbf{x}) - sh\,\beta\,\mathbf{e}\,(\mathbf{e}\,sh\,\beta\,t + \mathbf{x} + (ch\,\beta - 1)\,\mathbf{e}\,(\mathbf{ex})) = \\ = t + ch\,\beta\,sh\,\beta\,(\mathbf{e}\,\mathbf{x}) - sh\,\beta\,\mathbf{e}\,[\,\mathbf{x} + (ch\,\beta - 1)\,\mathbf{e}\,(\mathbf{ex})\,] = \\ = t + ch\,\beta\,sh\,\beta\,(\mathbf{e}\,\mathbf{x}) - sh\,\beta\,(\mathbf{e}\,\mathbf{x}) - sh\,\beta\;\;ch\,\beta(\mathbf{ex}) + sh\,\beta\;\;(\mathbf{ex}) = t\,. \end{aligned}$$

In the same manner, for the space variable $\mathbf{x}$ we have

$$\begin{aligned} \mathbf{x} = -\mathbf{e}\,sh\,\beta\,t' + \mathbf{x}' + (ch\,\beta - 1)\,\mathbf{e}\,(\mathbf{ex}') = \\ = -\mathbf{e}\,sh\,\beta\,[\,ch\,\beta\,t + sh\,\beta\,(\mathbf{e}\,\mathbf{x})\,] + \\ + \mathbf{e}\,sh\,\beta\,t + \mathbf{x} + (ch\,\beta - 1)\,\mathbf{e}\,(\mathbf{ex}) + \\ + (ch\,\beta - 1)\,\mathbf{e}\,[\,\mathbf{e}\,[\,(\mathbf{e}\,sh\,\beta\,t + \mathbf{x} + (ch\,\beta - 1)\,\mathbf{e}\,(\mathbf{ex})\,]\,] = \\ = -\mathbf{e}\,sh\,\beta\,ch\,\beta\,t - \;sh^2\beta\mathbf{e}\,(\mathbf{e}\,\mathbf{x}) + \mathbf{e}\,sh\,\beta\,t + \mathbf{x} + (ch\,\beta - 1)\,\mathbf{e}\,(\mathbf{ex}) + \\ + (ch\,\beta - 1)\,\mathbf{e}\,[\,(\;sh\,\beta\,t + ch\,\beta(\mathbf{ex})] = \\ = \mathbf{x} + [\,-sh^2\beta + ch\,\beta - 1 + ch^2\,\beta - ch\,\beta\,]\,\mathbf{e}\,(\mathbf{e}\,\mathbf{x}) = \mathbf{x}\,. \end{aligned}$$

It can be shown by straightforward calculation that the general Lorentz transformation (81) leaves invariant the so called relativistic length of 4-vector $(t, \mathbf{x})$:

$$\begin{aligned} t'^2 - \mathbf{x}'^2 = [\,ch\,\beta\,t + sh\,\beta\,(\mathbf{e}\,\mathbf{x})\,]^2 - \\ - [\,\mathbf{e}\,sh\,\beta\,t + \mathbf{x} + (ch\,\beta - 1)\,\mathbf{e}\,(\mathbf{ex})\,]^2 = \\ = ch^2\beta\,t^2 + 2ch\,\beta\,sh\,\beta\,t(\mathbf{e}\,\mathbf{x}) + sh^2\beta\,(\mathbf{e}\,\mathbf{x})^2 - \\ - sh^2\beta\,t^2 - 2sh\,\beta t\,(\mathbf{ex}) - \mathbf{x}^2\; - \\ - 2sh\,\beta(ch\,\beta - 1)t\,(\mathbf{ex}) - 2(ch\,\beta - 1)\,(\mathbf{ex})^2 - \\ - (ch\,\beta - 1)^2\,(\mathbf{ex})^2 = \\ = (t^2 - \mathbf{x}^2) + [\,2ch\,\beta\,sh\,\beta\,t(\mathbf{e}\,\mathbf{x}) + sh^2\beta\,(\mathbf{e}\,\mathbf{x})^2 - 2sh\,\beta t\,(\mathbf{ex}) - \\ - 2sh\,\beta ch\,\beta t'\,(\mathbf{ex}') + 2sh\,\beta\,t'\,(\mathbf{ex}') - 2(ch\,\beta - 1)\,(\mathbf{ex}')^2 - (ch\,\beta - 1)^2\,(\mathbf{ex}')^2 = \\ = (t'^2 - \mathbf{x}'^2) + [\,sh^2\beta\;\; - 2(ch\,\beta - 1) - (ch\,\beta - 1)^2]\,(\mathbf{e}\,\mathbf{x}')^2 = \\ = (t'^2 - \mathbf{x}'^2) + [\,sh^2\beta\;\; - 2ch\,\beta + 2 - ch\,^2\beta + 2ch\,\beta - 1]\,(\mathbf{e}\,\mathbf{x}')^2 = \\ = (t'^2 - \mathbf{x}'^2) + 0\,, \end{aligned}$$

so that

$$(t^2 - \mathbf{x}^2) = (t^{'2} - \mathbf{x}^{'2}) \; .$$

The formulas (81) can be rewritten differently with special notation for longitudinal and perpendicular constituents:

$$\mathbf{e}\,(\mathbf{e}\mathbf{x}') = \mathbf{x}'_{\parallel} \; , \qquad (\mathbf{e}\mathbf{x}') = x'_{\parallel}, \qquad \mathbf{x}' - \mathbf{e}\,(\mathbf{e}\mathbf{x}') = \mathbf{x}'_{\perp} \tag{82}$$

then

$$t = ch\;\beta\;t' + sh\;\beta\;x'_{\parallel} \; ,$$
$$\mathbf{x} = \mathbf{e}(\;sh\;\beta\;t' + ch\;\beta\;x'_{\parallel}) \;+\; \mathbf{x}'_{\perp} \; ,$$

or

$$t = ch\;\beta\;t' + sh\;\beta\;x'_{\parallel} \; , \qquad \mathbf{x}_{\perp} = \mathbf{x}'_{\perp} \; ,$$
$$\mathbf{x}_{\parallel} = \mathbf{e}\,(\mathbf{e}\mathbf{x}) = \mathbf{e}(sh\;\beta\;t' + ch\;\beta\;x'_{\parallel}) \; . \tag{83}$$

Now, with general expression for Lorentz transform in hands, we may quite easily obtain a general form of velocity addition law. Let us take two events on the trajectory of a moving particle

$$(t'_1, \mathbf{x}'_1) \qquad \text{and} \qquad (t'_2, \mathbf{x}'_2) \; ;$$

the velocity vector in the reference frame K' is defined by the relation

$$\mathbf{W}' = \frac{\mathbf{x}'_2 - \mathbf{x}'_1}{t'_2 - t'_1} \; ; .$$

In the same manner, the velocity vector in the reference frame K is given by

$$\mathbf{W} = \frac{\mathbf{x}_2 - \mathbf{x}_1}{t_2 - t_1} =$$
$$= \frac{-\mathbf{e}\;sh\;\beta\;t'_2 + \mathbf{x}'_2 + (ch\;\beta - 1)\;\mathbf{e}\;(\mathbf{e}\mathbf{x}'_2) + \mathbf{e}\;sh\;\beta\;t'_1 - \mathbf{x}'_1 - (ch\;\beta - 1)\;\mathbf{e}\;(\mathbf{e}\mathbf{x}'_1)}{ch\;\beta\;t'_2 - sh\;\beta\;\mathbf{e}\;\mathbf{x}'_2 - ch\;\beta\;t'_1 + sh\;\beta\;\mathbf{e}\;\mathbf{x}'_1} =$$
$$= \frac{(\mathbf{x}'_2 - \mathbf{x}'_1) - \mathbf{e}\;sh\;\beta\;(t'_2 - t'_1) + (ch\;\beta - 1)\;\mathbf{e}\;[\mathbf{e}(\mathbf{x}'_2 - \mathbf{x}'_1)]}{ch\;\beta\;(t'_2 - t'_1) - sh\;\beta\;\mathbf{e}\;(\mathbf{x}'_2 - \mathbf{x}'_1)} \; ;$$

from this it follows the velocity addition rule in the most general form

$$\mathbf{W} = \frac{\mathbf{W}' - \mathbf{e}\;sh\;\beta + (ch\;\beta - 1)\;\mathbf{e}(\mathbf{e}\;\mathbf{W}')}{ch\;\beta - sh\;\beta\;\mathbf{e}\;\mathbf{W}'} \; . \tag{84}$$

Eq. (84) may be rewritten as

$$\mathbf{W} = \frac{\mathbf{W}' - \mathbf{e}(\mathbf{e}\;\mathbf{W}')}{ch\;\beta - sh\;\beta\;\mathbf{e}\;\mathbf{W}'} \;+\; \frac{ch\;\beta\;\mathbf{e}(\mathbf{e}\;\mathbf{W}') - \mathbf{e}\;sh\;\beta}{ch\;\beta - sh\;\beta\;\mathbf{e}\;\mathbf{W}'} \; ,$$

which, with the notation

$$\mathbf{e}(\mathbf{e}\,\mathbf{W}') = \mathbf{W}'_{\|}, \qquad \mathbf{W}' - \mathbf{e}(\mathbf{e}\,\mathbf{W}') = \mathbf{W}'_{\perp}\ ,$$

will give

$$\mathbf{W} = \frac{\sqrt{1-V^2}}{1-\mathbf{W}'\mathbf{V}}\ \mathbf{W}'_{\perp} \ + \ \frac{\mathbf{W}'_{\|} - \mathbf{V}}{1-\mathbf{W}'\mathbf{V}}\ . \tag{85}$$

Evidently, eqs. (85) and (84) are equivalent to each other: eq. (85) seems more understandable from physical standpoint, though eq. (84) is more convenient at general calculating.

9 Reflection of the light in moving reference frame K, the case of an arbitrary velocity vector $\mathbf{V}$

Three vectors in the reflection law (see (70))

$$\mathbf{a}' \times \mathbf{n}' = \mathbf{b}' \times \mathbf{n}' \tag{86}$$

change at Lorentz transform $L(\mathbf{V})$ as follows

$$\mathbf{a}' = \frac{\mathbf{a} + \mathbf{e}\,[\ sh\ \beta + (ch\ \beta - 1)\ (\mathbf{e}\ \mathbf{a})\ ]}{ch\ \beta + sh\ \beta\ (\mathbf{ea})}\ ,$$
$$\mathbf{b}' = \frac{\mathbf{b} + \mathbf{e}\,[\ sh\ \beta + (ch\ \beta - 1)\ (\mathbf{e}\ \mathbf{b})\ ]}{ch\ \beta + sh\ \beta\ (\mathbf{eb})}\ ,$$
$$\mathbf{n}' = \frac{\mathbf{n} + \mathbf{e}\,[\ sh\ \beta + (ch\ \beta - 1)\ (\mathbf{e}\ \mathbf{n})\ ]}{ch\ \beta + sh\ \beta\ (\mathbf{en})}\ . \tag{87}$$

Substituting eqs. (87) into (86) we get to

$$\frac{\mathbf{a} + \mathbf{e}\,[\ sh\ \beta + (ch\ \beta - 1)\ (\mathbf{e}\ \mathbf{a})\ ]}{ch\ \beta - sh\ \beta\ \mathbf{e}\ \mathbf{a}} \times \frac{\mathbf{n} + \mathbf{e}\,[\ sh\ \beta + (ch\ \beta - 1)\ (\mathbf{e}\ \mathbf{n})\ ]}{ch\ \beta + sh\ (\beta\mathbf{en})} =$$
$$= \frac{\mathbf{b} + \mathbf{e}\,[\ sh\ \beta + (ch\ \beta - 1)\ (\mathbf{e}\ \mathbf{b})\ ]}{ch\ \beta + sh\ \beta\ \mathbf{e}\ \mathbf{b}} \times \frac{\mathbf{n} + \mathbf{e}\,[\ sh\ \beta + (ch\ \beta - 1)\ (\mathbf{e}\ \mathbf{n})\ ]}{ch\ \beta + sh\ (\beta\mathbf{en})}$$

and further

$$\frac{\mathbf{a}\times\mathbf{n} + (\mathbf{a}\times\mathbf{e})\,[\ sh\ \beta + (ch\ \beta - 1)\ (\mathbf{en})\ ] + (\mathbf{e}\times\mathbf{n})\,[\ sh\ \beta + (ch\ \beta - 1)\ (\mathbf{ea})\ ]}{ch\ \beta + sh\ \beta(\mathbf{ea})} =$$
$$= \frac{\mathbf{b}\times\mathbf{n} + (\mathbf{b}\times\mathbf{e})\,[\ sh\ \beta + (ch\ \beta - 1)\ (\mathbf{en})\ ] + (\mathbf{e}\times\mathbf{n})\,[\ sh\ \beta + (ch\ \beta - 1)\ (\mathbf{eb})\ ]}{ch\ \beta + sh\ \beta(\mathbf{eb})}\ .$$

After simple manipulation it gives

$$\frac{\mathbf{a}\times\mathbf{n} + (\mathbf{a}\times\mathbf{e} + \mathbf{e}\times\mathbf{n})\ sh\ \beta + [(\mathbf{a}\times\mathbf{e})\ (\mathbf{en}) + (\mathbf{e}\times\mathbf{n})\ (\mathbf{ea})\ ](ch\ \beta - 1)}{ch\ \beta + sh\ \beta(\mathbf{ea})} =$$
$$\frac{\mathbf{b}\times\mathbf{n} + (\mathbf{b}\times\mathbf{e} + \mathbf{e}\times\mathbf{n})\ sh\ \beta + [(\mathbf{b}\times\mathbf{e})\ (\mathbf{en}) + (\mathbf{e}\times\mathbf{n})\ (\mathbf{eb})\ ](ch\ \beta - 1)}{ch\ \beta + sh\ \beta(\mathbf{eb})}\ . \tag{88}$$

Taking the known relations for double vector product:

$$\mathbf{e} \times (\mathbf{n} \times \mathbf{a}) = \mathbf{n}(\mathbf{ea}) - \mathbf{a}(\mathbf{en}) \;,$$
$$\mathbf{e} \times (\mathbf{n} \times \mathbf{b}) = \mathbf{n}(\mathbf{eb}) - \mathbf{b}(\mathbf{en}) \;,$$

eq. (88) may be taken to the form

$$\frac{\mathbf{a} \times \mathbf{n} + (\mathbf{a} \times \mathbf{e} + \mathbf{e} \times \mathbf{n})\; sh\,\beta + \mathbf{e} \times (\mathbf{e} \times (\mathbf{n} \times \mathbf{a}))(ch\,\beta - 1)}{ch\,\beta + sh\,\beta\;(\mathbf{ea})} =$$
$$= \frac{\mathbf{b} \times \mathbf{n} + (\mathbf{b} \times \mathbf{e} + \mathbf{e} \times \mathbf{n})\; sh\,\beta + \mathbf{e} \times (\mathbf{e} \times (\mathbf{n} \times \mathbf{b}))(ch\,\beta - 1)}{ch\,\beta + sh\,\beta\;(\mathbf{eb})} \;. \tag{89}$$

Again using the known identities

$$\mathbf{e} \times (\mathbf{e} \times (\mathbf{n} \times \mathbf{a})) = \mathbf{e}\,[\,\mathbf{e}(\mathbf{n} \times \mathbf{a})\,] - (\mathbf{n} \times \mathbf{a}) \;,$$
$$\mathbf{e} \times (\mathbf{e} \times (\mathbf{n} \times \mathbf{b})) = \mathbf{e}\,[\,\mathbf{e}(\mathbf{n} \times \mathbf{b})\,] - (\mathbf{n} \times \mathbf{a}) \;,$$

eq. (89) will read

$$\frac{ch\,\beta\;(\mathbf{a} \times \mathbf{n}) + sh\,\beta\;(\mathbf{a} - \mathbf{n}) \times \mathbf{e} + (ch\,\beta - 1)\;\mathbf{e}\,[\,\mathbf{e}(\mathbf{n} \times \mathbf{a})\,]}{ch\,\beta + sh\,\beta\;(\mathbf{ea})} =$$
$$= \frac{ch\,\beta\;(\mathbf{b} \times \mathbf{n}) + sh\,\beta(\mathbf{b} - \mathbf{n}) \times \mathbf{e} + (ch\,\beta - 1)\;\mathbf{e}\,[\,\mathbf{e}(\mathbf{n} \times \mathbf{b})\,]}{ch\,\beta + sh\,\beta\;(\mathbf{eb})} \;,$$

or

$$\frac{\mathbf{a} \times \mathbf{n} + th\,\beta\;(\mathbf{a} - \mathbf{n}) \times \mathbf{e} + (1 - ch^{-1}\beta)\;\mathbf{e}\,[\,\mathbf{e}(\mathbf{n} \times \mathbf{a})\,]}{1 + th\,\beta\;(\mathbf{ea})} =$$
$$= \frac{\mathbf{b} \times \mathbf{n} + th\,\beta(\mathbf{b} - \mathbf{n}) \times \mathbf{e} + (1 - ch^{-1}\beta)\;\mathbf{e}\,[\,\mathbf{e}(\mathbf{n} \times \mathbf{b})\,]}{1 + th\,\beta\;(\mathbf{eb})} \;. \tag{90}$$

These equations provides us with mathematical form
of the reflection law in the moving reference frame K.

One may note that if four vector $\mathbf{a}, \mathbf{b}, \mathbf{n}, \mathbf{e}$ belong the same plane (previously obtained eq. (74) relates to just that situation) then two identities hold

$$[\,\mathbf{e}(\mathbf{n} \times \mathbf{a})\,] = 0, \qquad [\,\mathbf{e}(\mathbf{n} \times \mathbf{b})\,] = 0 \;, \tag{91}$$

at this eq. (90) will become much simpler;

$$\frac{\mathbf{a} \times \mathbf{n} + th\,\beta(\mathbf{a} - \mathbf{n}) \times \mathbf{e}}{1 + th\,\beta\;(\mathbf{ea})} = \frac{\mathbf{b} \times \mathbf{n} + th\,\beta\;(\mathbf{b} - \mathbf{n}) \times \mathbf{e}}{1 + th\,\beta\;(\mathbf{eb})} \;, \tag{92}$$

what coincides with eq. (74):

$$\frac{\mathbf{a} \times \mathbf{n} + (\mathbf{a} - \mathbf{n}) \times \mathbf{V}}{1 + \mathbf{aV}} = \frac{\mathbf{b} \times \mathbf{n} + (\mathbf{b} - \mathbf{n}) \times \mathbf{V}}{1 + \mathbf{bV}} \;.$$

Now let us perform some additional calculation that permit us to decompose the vector form of the reflection law (90) into two more simple equations: one along the $\mathbf{e}$ vector and other transversely to $\mathbf{e}$. Indeed, forming the dot product of eq (90) and the vector $\mathbf{e}$ we obtain

$$\frac{\mathbf{e}\,(\mathbf{a}\times\mathbf{n}) + th\,\beta\,\mathbf{e}\,[\,(\mathbf{a}-\mathbf{n})\times\mathbf{e}\,] + (1-ch^{-1}\beta)\,[\,\mathbf{e}(\mathbf{n}\times\mathbf{a})\,]}{1+th\,\beta\,(\mathbf{ea})} =$$
$$= \frac{\mathbf{e}\,(\mathbf{b}\times\mathbf{n}) + th\,\beta\mathbf{e}\,[\,(\mathbf{b}-\mathbf{n})\times\mathbf{e}\,] + (1-ch^{-1}\beta)\,[\,\mathbf{e}(\mathbf{n}\times\mathbf{b})\,]}{1+th\,\beta\,(\mathbf{eb})}\,.$$

from this it follows

$$(1-ch^{-1}\beta)\,\frac{[\mathbf{e}(\mathbf{n}\times\mathbf{a})]}{1+th\,\beta\,(\mathbf{ea})} = (1-ch^{-1}\beta)\,\frac{[\mathbf{e}(\mathbf{n}\times\mathbf{b})]}{1+th\,\beta\,(\mathbf{eb})}\,,$$

and further

$$\frac{\mathbf{V}(\mathbf{n}\times\mathbf{a})}{1+(\mathbf{Va})} = \frac{\mathbf{V}(\mathbf{n}\times\mathbf{b})}{1+(\mathbf{Vb})}\,. \qquad (93)$$

Taking this in mind , we see that in eq. (90) some terms on the left and on the right canceled each other so we will obtain (92). More clarity may be reached with the use of symbolical vector notation for eq. (90):

$$\mathbf{A} = \mathbf{B}\,.$$

The above calculation consists of two step:

$$\underline{first} \qquad \mathbf{eA} = \mathbf{eB}\,, \qquad \underline{second} \qquad \mathbf{A} - \mathbf{e}(\mathbf{eA}) = \mathbf{B} - \mathbf{e}(\mathbf{eB})\;;.$$

So, the light reflection law (90) in the moving reference frame K may be presented with the help of two equations

$$\frac{[\mathbf{V}(\mathbf{n}\times\mathbf{a})]}{1+(\mathbf{Va})} = \frac{[\mathbf{V}(\mathbf{n}\times\mathbf{b})]}{1+(\mathbf{Vb})}\,,$$
$$\frac{\mathbf{a}\times\mathbf{n} + (\mathbf{a}-\mathbf{n})\times\mathbf{V}}{1+\mathbf{aV}} = \frac{\mathbf{b}\times\mathbf{n} + (\mathbf{b}-\mathbf{n})\times\mathbf{V}}{1+\mathbf{bV}}\,. \qquad (94)$$

Now let us consider second (scalar) equation (71)

$$\mathbf{a}'\mathbf{n}' + \mathbf{b}'\mathbf{n}' = 0\,. \qquad (95)$$

From (95), with the use of (87), one gets

$$\frac{\mathbf{a} + \mathbf{e}\,[\,sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e\,a})\,]}{ch\,\beta + sh\,\beta\,(\mathbf{ea})}\;\frac{\mathbf{n} + \mathbf{e}\,[\,sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e\,n})\,]}{ch\,\beta + sh\,\beta\,(\mathbf{en})} +$$
$$+\frac{\mathbf{b} + \mathbf{e}\,[\,sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e\,b})\,]}{ch\,\beta + sh\,\beta\,(\mathbf{eb})}\;\frac{\mathbf{n} + \mathbf{e}\,[\,sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e\,n})\,]}{ch\,\beta + sh\,\beta\,(\mathbf{en})} = 0\,;$$

from where after evident calculation it follows

$$\frac{\mathbf{an} + (\mathbf{a}+\mathbf{n})\mathbf{e} sh\beta + 2(\mathbf{ae})(\mathbf{ne})(ch\beta - 1) + [sh\beta + (ch\beta-1)\mathbf{en}][sh\beta + (ch\beta-1)\mathbf{ea}]}{ch\beta + sh\beta\ \mathbf{ea}} +$$
$$+\frac{\mathbf{bn} + (\mathbf{b}+\mathbf{n})\mathbf{e}) sh\beta + 2(\mathbf{be})(\mathbf{ne})(ch\beta - 1) + [sh\beta + (ch\beta-1)\mathbf{en}][sh\beta + (ch\beta-1)\mathbf{eb}]}{ch\beta + sh\beta\ \mathbf{eb}} = 0$$

and further

$$\frac{\mathbf{an} + (\mathbf{a}+\mathbf{n})\mathbf{e}\ sh\beta ch\beta + [\ 1 + (\mathbf{en})(\mathbf{ea})\ ]\ sh^2\beta}{ch\beta + sh\beta \mathbf{ea}} +$$
$$+\frac{\mathbf{bn} + (\mathbf{b}+\mathbf{n})\mathbf{e}\ sh\beta ch\beta + [\ 1 + (\mathbf{en})(\mathbf{eb})\ ]\ sh^2\beta}{ch\beta + sh\beta \mathbf{eb}} = 0\ . \qquad (96)$$

Let us verify that this general formula (96) is reduced to the established result for a particular simpler case. Indeed. let it be

$$\mathbf{e} = (1,0,0), \qquad \mathbf{a} = (a_1, a_2, a_3), \qquad \mathbf{b} = (b_1, b_2, b_3)$$

then eq. (96) reads

$$\frac{(a_1 n_1 + a_2 n_2 + a_3 n_3)(ch^2\beta - sh^2\beta) + (a_1 + n_1) sh\beta ch\beta + (1 + a_1 n_1)\ sh^2\beta}{ch\beta + sh\beta a_1} +$$
$$+\frac{(b_1 n_1 + b_2 n_2 + b_3 n_3)(ch^2\beta - sh^2\beta) + (b_1 + n_1) sh\beta ch\beta + (1 + b_1 n_1)\ sh^2\beta}{ch\beta + sh\beta b_1} = 0$$

from where it follows

$$\frac{(a_1 n_1 + a_2 n_2 + a_3 n_3) ch^2\beta + (a_1 + n_1) sh\beta ch\beta + (1 - a_2 n_2 - a_3 n_3)\ sh^2\beta}{ch\beta + sh\beta a_1} +$$
$$+\frac{(b_1 n_1 + b_2 n_2 + b_3 n_3) ch^2\beta + (b_1 + n_1) sh\beta ch\beta + (1 - b_2 n_2 - b_3 n_3)\ sh^2\beta}{ch\beta + sh\beta b_1} = 0$$

which after dividing by $ch^2\beta$ gives the yet known result (77):

$$\frac{(a_1 n_1 + a_2 n_2 + a_3 n_3) + (a_1 + n_1)V + V^2(1 - a_2 n_2 - a_3 n_3)}{1 + a_1 V} +$$
$$+\frac{(b_1 n_1 + b_2 n_2 + b_3 n_3) + (b_1 + n_1)V + V^2(1 - b_2 n_2 - b_3 n_3)}{1 + b_1 V} = 0\ . \qquad (97)$$

Bearing in mind the previous analysis, eq. (96) can readily be changed to the form

$$\frac{\mathbf{an}(1 - V^2) + (\mathbf{a}+\mathbf{n})\mathbf{V} + [\ V^2 + (\mathbf{aV})(\mathbf{nV})\ ]}{1 + \mathbf{aV}} +$$
$$+\frac{\mathbf{bn} + (\mathbf{b}+\mathbf{n})\mathbf{V} + [\ V^2 + (\mathbf{bV})(\mathbf{nV})\ ]}{1 + \mathbf{bV}} = 0\ ; \qquad (98)$$

eq. (98) is the result of converting eq. (95) to moving reference frame.

Now let us give special attention to one other aspect of the problem under consideration. In the starting (rest) reference frame K' three vectors $\mathbf{a}, \mathbf{n}, \mathbf{b}$ belong to the same single plane, as a

result one can derive a decomposition of $\mathbf{b}$ into a linear combination of two remaining vectors: see Fig. 6.

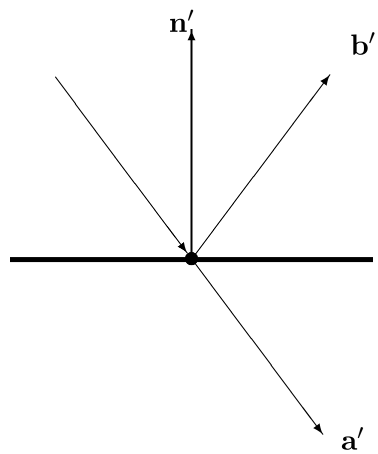


Fig. 6 The plane of incident, reflected, and normal rays

Let it be

$$\mathbf{b}' = \alpha\, \mathbf{a}' + \nu\, \mathbf{n}'.$$

With the use of

$$\mathbf{a}' \times \mathbf{n}' = (\alpha\, \mathbf{a}' + \nu\, \mathbf{n}') \times \mathbf{n}', \qquad \Longrightarrow \qquad \alpha = +1\ ,\ \mathbf{b}' = \mathbf{a}' + \nu\, \mathbf{n}'.$$

and

$$(\mathbf{a}' + \mathbf{b})'\mathbf{n}' = 0\ , \qquad \Longrightarrow \qquad (2\mathbf{a}' + \nu\mathbf{n}')\, \mathbf{n}' = 0\ ,$$

we get to

$$\nu = -2(\mathbf{a}'\mathbf{n}') = -2\cos\phi\ .$$

Therefore, in the rest reference frame K'
the reflection law can be formulated as follows

$$\mathbf{b}' = \mathbf{a}' - 2(\mathbf{a}'\mathbf{n}')\, \mathbf{n}'\ . \tag{99}$$

We are to convert this equation to the moving reference frame K:

$$\frac{\mathbf{b} + \mathbf{e}\,[\ sh\,\beta + (ch\,\beta - 1)\ (\mathbf{e\,b})\ ]}{ch\,\beta + sh\,\beta\ (\mathbf{eb})} = \frac{\mathbf{a} + \mathbf{e}\,[\ sh\,\beta + (ch\,\beta - 1)\ (\mathbf{e\,a})\ ]}{ch\,\beta + sh\,\beta\ (\mathbf{ea})} -$$
$$-2\left(\frac{\mathbf{a} + \mathbf{e}\,[\ sh\,\beta + (ch\,\beta - 1)\ (\mathbf{e\,a})\ ]}{ch\,\beta + sh\,\beta\ (\mathbf{ea})}\ \frac{\mathbf{n} + \mathbf{e}\,[\ sh\,\beta + (ch\,\beta - 1)\ (\mathbf{e\,n})\ ]}{ch\,\beta + sh\,\beta\ (\mathbf{en})}\right)$$
$$(\times)\ \frac{\mathbf{n} + \mathbf{e}\,[\ sh\,\beta + (ch\,\beta - 1)\ (\mathbf{e\,n})\ ]}{ch\,\beta + sh\,\beta\ (\mathbf{en})}\ ;$$

or in a more short form

$$\frac{\mathbf{b} + \mathbf{e}\,[\, sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{b})\,]}{ch\,\beta + sh\,\beta\,(\mathbf{eb})} =$$

$$= \frac{\mathbf{a} + \mathbf{e}\,[\, sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{a})\,]}{ch\,\beta + sh\,\beta\,(\mathbf{ea})} + \nu\,\frac{\mathbf{n} + \mathbf{e}\,[\, sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{n})\,]}{ch\,\beta + sh\,\beta\,(\mathbf{en})}\,. \tag{100}$$

It should be especially noted that the general structure of the relation obtained is

$$\mathbf{b} = A\,\mathbf{a} + B\,\mathbf{n} + C\,\mathbf{e}\,;$$

we see that there appears a constituent along $\mathbf{e}$ direction which means that the vector $\mathbf{b}$ (reflected light ray) is not situated in the plane of vectors $\mathbf{a}$ and $\mathbf{n}$.

In general, eq. (100) can be resolved with respect to $\mathbf{b}$. To this end, let us form the dot product of eq. (100) and vector $\mathbf{e}$:

$$\frac{sh\,\beta + ch\,\beta(\mathbf{e}\,\mathbf{b})}{ch\,\beta + sh\,\beta\,(\mathbf{eb})} = \frac{sh\,\beta + ch\,\beta(\mathbf{e}\,\mathbf{a})}{ch\,\beta + sh\,\beta\,(\mathbf{ea})} + \nu\,\frac{sh\,\beta + ch\,\beta(\mathbf{en})\,]}{ch\,\beta + sh\,\beta(\mathbf{en})} \equiv B\,; \tag{101}$$

the right hand side of eq. (101) is designated by B. From eq. (101) it follows

$$sh\,\beta + ch\,\beta(\mathbf{e}\,\mathbf{b}) = B\,ch\,\beta + B\,sh\,\beta\,(\mathbf{eb})\,, \qquad \Longrightarrow \qquad \mathbf{eb} = \frac{B - th\,\beta}{1 - B\,th\,\beta}\,. \tag{102}$$

The relation (101) is in accordance with the identity $B \equiv \mathbf{b}'\mathbf{e}$. Besides, taking into account eq. (101), the previous relation (100) can be much simplified:

$$\frac{\mathbf{b} - \mathbf{e}\,(\mathbf{e}\,\mathbf{b})}{ch\,\beta + sh\,\beta\,(\mathbf{eb})} = \frac{\mathbf{a} - \mathbf{e}\,(\mathbf{e}\,\mathbf{a})}{ch\,\beta + sh\,\beta\,(\mathbf{ea})} + \nu\,\frac{\mathbf{n} - \mathbf{e}\,(\mathbf{e}\,\mathbf{n})\,]}{ch\,\beta + sh\,\beta\,(\mathbf{en})}$$

or

$$\frac{\mathbf{b}_\perp}{1 + \mathbf{bV}} = \frac{\mathbf{a}_\perp}{1 + \mathbf{aV}} + \nu\,\frac{\mathbf{n}_\perp}{1 + \mathbf{nV}}\,. \tag{103}$$

It is quite understandable that (101) and (103) are the projections of eq. (100) onto direction of $\mathbf{e}$ and a plane orthogonal to it.

In the end let us calculate an additional characteristic of the reflection law, as sketched in Fig. 7:

$$\Delta' = \mathbf{n}'\,(\mathbf{a}' \times \mathbf{b}') = 0\,, \qquad \Longrightarrow \qquad \Delta = \mathbf{n}\,(\mathbf{a} \times \mathbf{b})\,.$$

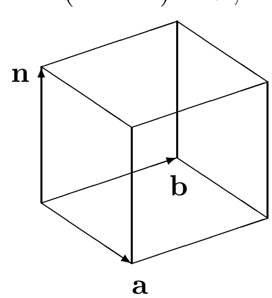


Fig 7. The parallelepiped of incident, normal, reflected rays

With the Lorentz formulas

$$\mathbf{a} = \frac{\mathbf{a}' + \mathbf{e}\,[\,-sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{a}')\,]}{ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{a}'}\ ,$$
$$\mathbf{b} = \frac{\mathbf{b}' + \mathbf{e}\,[\,-sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{b}')\,]}{ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{b}'}\ ,$$
$$\mathbf{n} = \frac{\mathbf{n} + \mathbf{e}\,[\,-sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{n}')\,]}{ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{n}'}$$

we get

$$\begin{aligned} \Delta = \frac{\mathbf{n}' + \mathbf{e}\,[\,-sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{n}')\,]}{ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{n}'} \bullet \\ \bullet \left(\frac{\mathbf{a}' + \mathbf{e}\,[\,-sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{a}')\,]}{ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{a}'} \times \frac{\mathbf{b}' + \mathbf{e}\,[\,-sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{b}')\,]}{ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{b}'} \right) = \\ = \frac{\mathbf{n}' + \mathbf{e}\,[\,-sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{n}')\,]}{ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{n}'} \bullet \\ \bullet \frac{\mathbf{a}' \times \mathbf{b}' + \mathbf{a}' \times \mathbf{e}\,[\,-sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{b}')\,] + \mathbf{e} \times \mathbf{b}'\,[\,-sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{a}')\,]}{(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{a}')(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{b}')}\ , \end{aligned}$$

and further

$$\begin{aligned} \Delta = \frac{1}{(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{n}')\,(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{a}')(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{b}')} \\ \{\ [\,\mathbf{n}'(\mathbf{a}' \times \mathbf{e})\,]\,[\,-sh\beta + (ch\beta - 1)\,\mathbf{e}\mathbf{b}'\,]\ + \\ +\,[\,\mathbf{n}'(\mathbf{e} \times \mathbf{b}')\,]\,[\,-sh\beta + (ch\beta - 1)\,\mathbf{e}\mathbf{a}'\,] + \\ +\,[\,\mathbf{e}(\mathbf{a}' \times \mathbf{b}')\,]\,[-sh\beta + (ch\beta - 1)\,\mathbf{e}\mathbf{n}'\,]\,\}\ . \end{aligned} \tag{104}$$

Bearing in mind eq. (99)

$$\mathbf{b}' = \mathbf{a}' - 2(\mathbf{a}'\mathbf{n}')\,\mathbf{n}' = \mathbf{a}' + \nu\,\mathbf{n}'\ ,$$

eq. (104) reads

$$\begin{aligned} \Delta = \frac{1}{(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{n}')\,(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{a}')\,(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{b}')}\,(\times) \\ \{\ [\,\mathbf{n}'(\mathbf{a}' \times \mathbf{e})\,]\,[\,-sh\beta + (ch\beta - 1)(\mathbf{e}\mathbf{a}' + \nu\,\mathbf{e}\mathbf{n}')\,]\ + \\ +\,[\,\mathbf{n}'(\mathbf{e} \times \mathbf{a}')\,]\,[\,-sh\beta + (ch\beta - 1)(\mathbf{e}\mathbf{a}')] + \\ +\,\nu\,[\,\mathbf{e}(\mathbf{a}' \times \mathbf{n}')\,]\,[-sh\beta + (ch\beta - 1)(\mathbf{e}\mathbf{n}')\,]\,\}\ . \end{aligned} \tag{105}$$

From this, with the help of the known symmetry properties of the mixed vector product, we produce

$$\begin{aligned} \Delta = \frac{[\,\mathbf{e}(\mathbf{n}' \times \mathbf{a}')\,]}{(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{n}')\,(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{a}')\,(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{b}')}\,(\times) \\ [\,-sh\beta + (ch\beta - 1)(\mathbf{e}\mathbf{a}' + \nu\,\mathbf{e}\mathbf{n}') + sh\beta - (ch\beta - 1)\,\mathbf{e}\mathbf{a}' + \nu sh\beta - \nu(ch\beta - 1)\,\mathbf{e}\mathbf{n}'\,] = \\ = \frac{\nu\,sh\,\beta\ \ [\,\mathbf{e}(\mathbf{n}' \times \mathbf{a}')\,]}{(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{n}')\,(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{a}')\,(ch\,\beta - sh\,\beta\,\mathbf{e}\mathbf{b}')}. \end{aligned}$$

So, finally we have obtained the result

$$\Delta = \mathbf{n}\,(\mathbf{a} \times \mathbf{b}) =$$
$$= \frac{-2\; sh\;\beta\;(\mathbf{a}'\mathbf{n}')\;[\;\mathbf{e}(\mathbf{n}' \times \mathbf{a}')\;]}{(ch\;\beta - sh\;\beta\;\mathbf{en}')\;(ch\;\beta - sh\;\beta\;\mathbf{ea}')\;(ch\;\beta - sh\;\beta\;\mathbf{eb}')}\;. \tag{106}$$

This relation gives the volumes of the light parallelepiped

$$[\;\mathbf{n}\,(\mathbf{a} \times \mathbf{b})\;]$$

in the moving reference frame as a function of the $(\beta, \mathbf{e};\ \mathbf{n}',\ \mathbf{a}')$ *. We must note that in contrast to the ordinary reflection law (when the expression* $A\mathbf{a}' + B\mathbf{n}' + C\mathbf{b}'$ *in fact presents a 2-dimensional object – plane) now in the moving reference frame* K *we will have in general a 3-space object* $A\mathbf{a} + B\mathbf{n} + C\mathbf{b}$ *.*

Now, one other aspect of the problem under consideration should be discussed. The obtained form of the light reflection law in the moving reference frame depends, strictly speaking, on the occasional choice of the direction of the normal light motion: namely, from the surface. However, one might chose opposite direction, that is the light ray going to the surface.

The matter is that the simple relation between two corresponding vectors

$$\mathbf{N}' = -\,\mathbf{n}'\;, \tag{107}$$

will not preserve its form with respect to Lorentz transformation. Indeed, in the moving reference frame we will have

$$\mathbf{n} = \frac{\mathbf{n}' + \mathbf{e}\;[\;-sh\;\beta + (ch\;\beta - 1)\;(\mathbf{e}\;\mathbf{n}')\;]}{ch\;\beta - sh\;\beta\;(\mathbf{en}')}\;,$$
$$\mathbf{N} = \frac{-\mathbf{n}' + \mathbf{e}\;[\;-sh\;\beta - (ch\;\beta - 1)\;(\mathbf{e}\;\mathbf{n}')\;]}{ch\;\beta + sh\;\beta\;(\mathbf{en}')}\;. \tag{108}$$

Two derivative characteristics can be calculated:

$$\mathbf{n} \times \mathbf{N} = \frac{2sh\;\beta}{ch^2\;\beta - sh^2\;\beta\;(\mathbf{en}')^2}\;\mathbf{e} \times \mathbf{n}\;, \qquad \mathbf{n} \bullet \mathbf{N} = 1 - \frac{2}{ch^2\;\beta - sh^2\beta(\mathbf{ed})^2}\;.$$

As it must be expected when $\beta = 0$ the equations give

$$\beta = 0 \qquad \Longrightarrow \qquad \mathbf{n} \times \mathbf{N} = 0\;,\;\; \mathbf{n} \bullet \mathbf{N} = -1\;.$$

It is readily verified the identity:

$$(\mathbf{n} \times \mathbf{N}) \bullet (\mathbf{n} \times \mathbf{N}) = 1 - (\mathbf{n} \bullet \mathbf{N})^2\;.$$

If, instead of the vector $\mathbf{n}$, one uses in formulating the light reflection law in the rest reference frame the opposite vector $\mathbf{N}$

$$\mathbf{a}' \times \mathbf{N}' = \mathbf{b}' \times \mathbf{N}'\;, \qquad \mathbf{a} \bullet \mathbf{N} + \mathbf{b} \bullet \mathbf{N} = 0\;, \tag{109}$$

then further in calculation no serious change will not appear: everywhere instead of $\mathbf{n}$ one will write another symbol $\mathbf{N}$. So that instead of (90) and (98) one has

$$\frac{ch\,\beta\,(\mathbf{a}\times\mathbf{N}) + sh\,\beta\,(\mathbf{a}-\mathbf{N})\times\mathbf{e} + (ch\,\beta - 1)\,\mathbf{e}\,[\,\mathbf{e}(\mathbf{N}\times\mathbf{a})\,]}{ch\,\beta + sh\,\beta\,(\mathbf{ea})} =$$
$$= \frac{ch\,\beta\,(\mathbf{b}\times\mathbf{N}) + sh\,\beta(\mathbf{b}-\mathbf{N})\times\mathbf{e} + (ch\,\beta - 1)\,\mathbf{e}\,[\,\mathbf{e}(\mathbf{N}\times\mathbf{b})\,]}{ch\,\beta + sh\,\beta\,(\mathbf{eb})}\,, \tag{110}$$

$$\frac{\mathbf{aN}(1-V^2) + (\mathbf{a}+\mathbf{N})\mathbf{V} + [\,V^2 + (\mathbf{aV})(\mathbf{NV})\,]}{1+\mathbf{aV}} +$$
$$+ \frac{\mathbf{bN} + (\mathbf{b}+\mathbf{N})\mathbf{V} + [\,V^2 + (\mathbf{bV})(\mathbf{NV})\,]}{1+\mathbf{bV}} = 0\,. \tag{111}$$

In turn, instead of (106) one derives

$$[\mathbf{N}\,(\mathbf{a}\times\mathbf{b})] =$$
$$= [\,\mathbf{e}(\mathbf{N}'\times\mathbf{a}')\,]\,\frac{-2\,sh\,\beta\,(\mathbf{a}'\mathbf{N}')}{(ch\,\beta - sh\,\beta\,\mathbf{eN}')\,(ch\,\beta - sh\,\beta\,\mathbf{ea}')\,(ch\,\beta - sh\,\beta\,\mathbf{eb}')} =$$
$$= [\,\mathbf{e}(\mathbf{n}'\times\mathbf{a}')\,]\,\frac{-2\,sh\,\beta\,(\mathbf{a}'\mathbf{n}')}{(ch\,\beta + sh\,\beta\,\mathbf{en}')\,(ch\,\beta - sh\,\beta\,\mathbf{ea}')\,(ch\,\beta - sh\,\beta\,\mathbf{eb}')}\,. \tag{112}$$

Existence of the two substantially different normal vectors in the moving reference frame makes us to consider the formulas obtained for the light reflection in the moving reference frame as ones describing the relativistic aberration effect for two different triples of light velocity vectors: $\mathbf{a},\mathbf{b},\mathbf{n}$ and $\mathbf{a},\mathbf{b},\mathbf{N}$. One may expect to reach more clarity in looking at the geometrical properties of the reflecting surfaces with respect to relativistic motion of the reference frame.

10 On the form of reflection surface in the moving reference frame

In the rest reference frame K', the reflection surface is a plane. Let $\mathbf{d}$ be its (unit) normal vector, the the known form of the plane is given by (here D is the distance from the origin O to the plane)

$$S' : \qquad \mathbf{x}'\,\mathbf{d}' + D = 0\,, \tag{113}$$

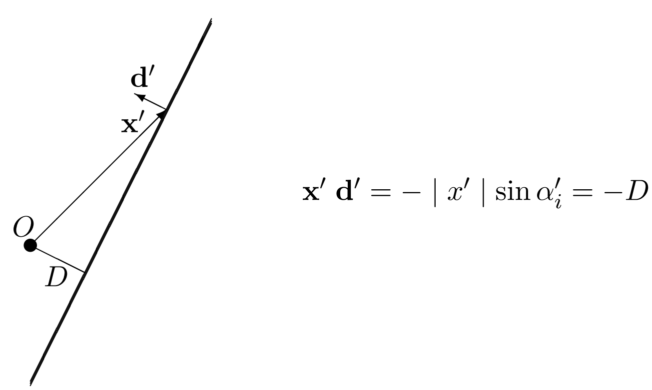


Fig. 8: The reflection plane in the rest reference frame K'

Now we should associate with this plane S', purely geometrical object, a set of events in space-time:

$$S' : \qquad \text{the plane } (\mathbf{d}, D) \qquad \Longrightarrow \qquad \{ \ (t', \mathbf{x}' = (x_1', x_2', x_3') \ \} \ .$$

To this end, at the moment $t' = 0$ let light signals be sent at all directions to the plane – the arriving of any particular signal to the plane gives rise to an event:

$$\mathbf{x}' = \mathbf{W}' \, t', \qquad \Longrightarrow \qquad \mathbf{x}^{'2} = \mathbf{W}^{'2} \, t^{'2}, \qquad t' = \sqrt{\mathbf{x}'^2} \ . \tag{114}$$

Therefore, to the reflection plane corresponds the following set of space-time events:

$$S' : \qquad \Longrightarrow \qquad \Big\{ \ t' = \sqrt{\mathbf{x}'^2} \ , \ \mathbf{x}' \, \mathbf{d}' + D' = 0 \ \Big\} \ . \tag{115}$$

The same set of event can be observed in the moving reference frame:

$$\mathbf{x} = \mathbf{W} \, t, \qquad \Longrightarrow \qquad \mathbf{x}^2 = \mathbf{W}^2 \, t^2, \qquad t = \sqrt{\mathbf{x}^2} \ . \tag{116}$$

As a result, the above equation of the plane (115) will take a new form (take notice that here the Lorentz transform acts only onto $\mathbf{x}'$ whereas the two quantities $\mathbf{d}$ and D' are considered as given parameters – below we will omit the *prime* symbol $'$ at them)

$$\Big\{ \ t = \sqrt{\mathbf{x}^2} \ , \ [\ \mathbf{x} + \mathbf{e} \ (\ sh \, \beta \, t + (ch \, \beta - 1) \ (\mathbf{ex}) \ ) \ ] \ \mathbf{d} + D = 0 \ \Big\} \ , \tag{117}$$

To obtain an equation for geometrical surface in the reference frame K one should exclude the variable t with the help of the first relation in (117):

$$S : \qquad (\mathbf{xd}) + (\mathbf{ed}) \ \Big[\ sh \, \beta \ (\ \sqrt{\mathbf{x}^2} \ ) + (ch \, \beta - 1) \ (\mathbf{ex}) \ \Big] + D = 0 \ . \tag{118}$$

This equation describes the geometry of the reflection surface S in the moving reference frame K. In the rest reference frame K' (when $\beta = 0$) eq. (118) becomes an equation of the plane.

To avoid misunderstanding it should be noted one special case: when vector $\mathbf{e}$ is orthogonal to the normal vector $\mathbf{d}$ we have the identity $(\mathbf{ed}) = 0$ which means that in this case the plane in the rest reference frame does not change its geometrical shape in the moving reference frame:

$$S : \qquad (\mathbf{xd}) + D = 0 \ .$$

Let us check our calculation by seing form of this general result (118) for a particular case

$$\mathbf{x} = (x, y, 0) \ , \qquad \mathbf{d} = (\cos\alpha, \sin\alpha, 0) \ , \qquad \mathbf{e} = (1, 0, 0) \ , \tag{119}$$

at this eq. (118) gives

$$x \cos\alpha + y \sin\alpha + \cos\alpha \ [\ sh \, \beta \sqrt{x^2 + y^2} + (ch \, \beta - 1) x \ ] + D = 0 \ ,$$

and further

$$y \sin\alpha + \cos\alpha (sh \, \beta \sqrt{x^2 + y^2} + ch \, \beta \ x \ ) + D = 0 \ .$$

Dividing it by $\sin\alpha$, after simple regrouping all the terms we will obtain

$$y = -\frac{D}{\sin\alpha} - \frac{1}{\tan\alpha}\ \Big[\ ch\ \beta + sh\ \beta\sqrt{1+y^2/x^2} + \Big]\ x\ , \tag{120}$$

what coincides with the previously found equation in Section 5:

$$y = b + k\ \Big[\ ch\ \beta + sh\ \beta\ \sqrt{1+y^2/x^2}\ \Big]\ x\ .$$

It may be easily verified that eq. (118) corresponds to a second order surface. Indeed,

$$-\,(\mathbf{ed})\ sh\ \beta\ (\ \sqrt{\mathbf{x}^2}\ ) = (\mathbf{xd}) + (\mathbf{ed})(\mathbf{ex})\ (ch\ \beta - 1) + D\ ,$$

and further we get to

$$(\mathbf{ed})^2 sh^2\beta\ \mathbf{x}^2 = [(\mathbf{xd}) + (\mathbf{ed})(\mathbf{ex})\ (ch\ \beta - 1) + D]^2\ . \tag{121}$$

11 Canonical form of the reflection surface S in the K-frame

Now we are to perform analysis extending the Section 5: with the use of a special rotation in 3-space and then a special shift in 3-space the equation of the reflection surface S

$$(\mathbf{ed})^2 sh^2\beta\ \mathbf{x}^2 = [(\mathbf{xd}) + (\mathbf{ed})(\mathbf{ex})\ (ch\ \beta - 1) + D]^2 \tag{122}$$

is to be taken to a canonical form (supposedly it is a hyperboloid).

We need some fact on rotations in 3-space. An arbitrary rotation matrix can be given as follows (more details see in [86]):

$$\begin{array}{lll}
O_1{}^1 = 1 - 2(c_2^2 + c_3^2)\ , & O_1{}^2 = -2c_0c_3 + 2c_1c_2\ , & O_1{}^3 = +2c_0c_2 + 2c_1c_3\ , \\
O_2{}^1 = +2c_0c_3 + 2c_1c_2\ , & O_2{}^2 = 1 - 2(c_3^2 + c_1^2)\ , & O_2{}^3 = -2c_0c_1 + 2c_2c_3\ , \\
O_3{}^1 = -2c_0c_2 + 2c_1c_3\ , & O_3{}^2 = +2c_0c_1 + 2c_2c_3\ , & O_3{}^3 = 1 - 2(c_1^2 + c_2^2)\ ,
\end{array} \tag{123}$$

where parameters b_a obey the condition

$$c_0^2 + c_1^2 + c_2^2 + c_3^2 = +1\ .$$

The rotation matrix O may be presented in the form more convenient at further calculation:

$$O = \left|\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right| + 2c_0 \left|\begin{array}{ccc} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{array}\right| + \left|\begin{array}{ccc} -2(c_2^2 + c_3^2) & 2c_1c_2 & 2c_1c_3 \\ 2c_1c_2 & -2(c_3^2 + c_1^2) & 2c_2c_3 \\ 2c_1c_3 & 2c_2c_3 & -2(c_1^2 + c_2^2)\ . \end{array}\right|$$

With the notation

$$\mathbf{c}^{\times} = \left|\begin{array}{ccc} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{array}\right|\ ,$$

the matrix O reads as

$$O = I + 2c_0 \, \mathbf{c}^{\times} + 2(\mathbf{c}^{\times})^2 \, . \tag{124}$$

The variable c_0 can be excluded:

$$\mathbf{C} = \frac{\mathbf{c}}{c_0}, \qquad O = I + 2 \, \frac{\mathbf{C}^{\times} + (\mathbf{C}^{\times})^2}{1 + \mathbf{C}^2} \, , \tag{125}$$

eq. (125) provides us with the known formula by Gibbs that is very helpful in practical calculation[6]; in particular, the composition rule has the form

$$O(\mathbf{C}') \, O(\mathbf{C}) = O(\mathbf{C}''), \qquad \mathbf{C}'' = \frac{\mathbf{C}' + \mathbf{C} + \mathbf{C}' \times \mathbf{C}}{1 - \mathbf{C}'\mathbf{C}} \, . \tag{126}$$

It is readily verified the identity

$$\mathbf{c}^{\times} \, \mathbf{A} = \mathbf{c} \times \mathbf{A} \, ,$$

so that the matrix O acts on a vector in accordance with the rule

$$O(\mathbf{c})\mathbf{A} = \mathbf{A} + 2c_0 \, \mathbf{c} \times \mathbf{A} + 2\mathbf{c} \times (\mathbf{c} \times \mathbf{A}) \, . \tag{127}$$

Instead of four parameters c_a one can use an angular variable $\phi/2$ and a unit vector $\mathbf{o}$):

$$c_0 = \cos\frac{\phi}{2} \, , \qquad \mathbf{c} = \sin\frac{\phi}{2} \, \mathbf{o}, \qquad \mathbf{o}^2 = 1 \, , \tag{128}$$

then eq. (127) looks

$$O \, \mathbf{A} = \mathbf{A} + \sin\phi \, (\mathbf{o} \times \mathbf{A}) + (1 - \cos\phi) \, \mathbf{o} \times (\mathbf{o} \times \mathbf{A}) \, . \tag{129}$$

With the known relation

$$\mathbf{o} \times (\mathbf{o} \times \mathbf{A}) = \mathbf{o} \, (\mathbf{o}\mathbf{A}) - \underline{A} \, ,$$

eq. (129) becomes

$$O(\phi, \mathbf{o}) \, \mathbf{A} = \mathbf{A} + \sin\phi \, (\mathbf{o} \times \mathbf{A}) + (1 - \cos\phi) \, [\mathbf{o} \, (\mathbf{o}\mathbf{A}) - \mathbf{A}] \, .$$

Thus,

the final formula for an arbitrary 3-rotation is as follows:

$$O(\phi, \mathbf{o}) \, \mathbf{A} = \cos\phi \, \mathbf{A} + \sin\phi \, (\mathbf{o} \times \mathbf{A}) + (1 - \cos\phi) \, (\mathbf{o}\mathbf{A}) \, \mathbf{o} \, . \tag{130}$$

in the right side we have a linear decomposition of the vector in terms of those

$$\mathbf{o} \, , \qquad \mathbf{A}, \qquad (\mathbf{A} \times \mathbf{o}) \, .$$

[6]More details see in [.....].

There are three simple cases:

$$\underline{\phi, \ \mathbf{o} = (1,0,0)} \qquad O\mathbf{A} =$$

$$= \cos\phi \begin{vmatrix} A_1 \\ A_2 \\ A_3 \end{vmatrix} + \sin\phi \begin{vmatrix} 0 \\ -A_3 \\ +A_2 \end{vmatrix} + (1-\cos\phi) \begin{vmatrix} A_1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{vmatrix} \begin{vmatrix} A_1 \\ A_2 \\ A_3 \end{vmatrix} ,$$

$$\underline{\phi, \ \mathbf{o} = (0,1,0)} \qquad O\mathbf{A} =$$

$$= \cos\phi \begin{vmatrix} A_1 \\ A_2 \\ A_3 \end{vmatrix} + \sin\phi \begin{vmatrix} A_3 \\ 0 \\ -A_1 \end{vmatrix} + (1-\cos\phi) \begin{vmatrix} 0 \\ A_2 \\ 0 \end{vmatrix} = \begin{vmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{vmatrix} \begin{vmatrix} A_1 \\ A_2 \\ A_3 \end{vmatrix} ,$$

$$\underline{\phi, \ \mathbf{o} = (0,0,1)} \qquad O\mathbf{A} =$$

$$= \cos\phi \begin{vmatrix} A_1 \\ A_2 \\ A_3 \end{vmatrix} + \sin\phi \begin{vmatrix} -A_2 \\ +A_1 \\ 0 \end{vmatrix} + (1-\cos\phi) \begin{vmatrix} 0 \\ 0 \\ A_3 \end{vmatrix} = \begin{vmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_1 \\ A_2 \\ A_3 \end{vmatrix} .$$

We return to the equation (122) now written in the form

$$\mathbf{x}^2 = (\mathbf{x} \bullet \mathbf{f} + F)^2, \tag{131}$$

where

$$\mathbf{f} = \frac{\mathbf{d} + (ch\,\beta - 1)(\mathbf{ed})\mathbf{e}}{(\mathbf{ed})\ sh\,\beta} , \qquad F = \frac{D}{(\mathbf{ed})\ sh\,\beta} .$$

Let us introduce new (rotated) variables $\mathbf{X}'$:

$$\mathbf{X} = O\,\mathbf{x} , \qquad \mathbf{x} = O^{-1}\,\mathbf{X} , \tag{132}$$

parameters c_a will be determined below. Take notice that any rotation matrix obeys the so called orthogonality condition: $O^{-1} = \tilde{O}$. In the new variables, eq. (131) looks as

$$(O^{-1}\mathbf{X})^2 = (O^{-1}\mathbf{X} \bullet \mathbf{f} + F)^2,$$

or

$$(\mathbf{X})^2 = (\mathbf{X} \bullet O\mathbf{f} + F)^2. \tag{133}$$

It is the point to determine a rotation needed: let the identity hold

$$O(c)\,\mathbf{f} = (0,0,f) , \qquad f > 0 ; \tag{134}$$

such a rotation $O(c)$ provides us with a new Cartesian coordinate system (X,Y,Z) for which the axis Z is located along the vector $\mathbf{f}$. An explicit form of that rotation can be found quite easily[7] As a result, eq. (133) reads

$$X^2 + Y^2 + Z^2 = Z^2 f^2 + 2ZFf + F^2 ; \tag{135}$$

[7]Some details see below.

or after simple calculation

$$X^2 + Y^2 + (1-f^2) \left[Z^2 - 2Z \frac{Ff}{1-f^2} + \frac{F^2 f^2}{(1-f^2)^2} \right] = F^2 + \frac{F^2 f^2}{1-f^2} ,$$

and further

$$X^2 + Y^2 + (1-f^2) \left[Z - \frac{Ff}{1-f^2} \right]^2 = \frac{F^2}{1-f^2} .$$

So we have arrived at the equation for reflection surface in the moving reference frame:

$$\frac{1-f^2}{F^2} X^2 + \frac{1-f^2}{F^2} Y^2 + \frac{(1-f^2)^2}{F^2} \left[Z - \frac{Ff}{1-f^2} \right]^2 = 1 ; \tag{136}$$

It remains to add some details for $1-f^2$. With the help of:

$$\mathbf{f} = \frac{\mathbf{d} + (ch\,\beta - 1)(\mathbf{ed})\mathbf{e}}{(\mathbf{ed})\; sh\,\beta} , \qquad 1 - f^2 = 1 - \frac{1 + 2(ch\,\beta - 1)(\mathbf{ed})^2 + (ch\,\beta - 1)^2 (\mathbf{ed})^2}{(\mathbf{ed})^2\; sh^2\,\beta} =$$

$$= \frac{1}{(\mathbf{ed})^2\; sh^2\,\beta} \left[(\mathbf{ed})^2\; sh^2\,\beta - 1 - (\mathbf{ed})^2 \left[2\; ch\,\beta - 2 + ch^2\beta - 2\; ch\,\beta + 1 \right] \right] , \qquad \Longrightarrow$$

$$1 - f^2 = \frac{-1}{(\mathbf{ed})^2\; sh^2\,\beta} \tag{137}$$

and also remembering

$$F^2 = \frac{D^2}{(\mathbf{ed})^2\; sh^2\,\beta} ,$$

to eq. (136) can be given the form

$$-X^2 - Y^2 + \frac{(Z - Z_0)^2}{(\mathbf{ed})^2\; sh^2\beta} = D^2 , \tag{138}$$

which is a canonical equation for hyperboloid. Its symmetry axis is located along the vector

$$\mathbf{f} = \frac{\mathbf{d} + (ch\,\beta - 1)(\mathbf{ed})\mathbf{e}}{(\mathbf{ed})\; sh\,\beta} , \qquad F = \frac{D}{(\mathbf{ed})\; sh\,\beta} . \tag{139}$$

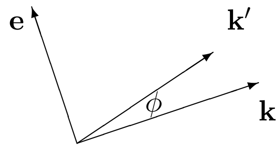
These formulas are correct only if $\mathbf{ed} \neq 0$.

Now let us briefly clarify how one can handle with a rotation relating two given vectors (of the same length)

$$\mathbf{K} = K\; \mathbf{k}, \qquad \mathbf{K}' = K\; \mathbf{k}', \qquad \mathbf{k}^2 = \mathbf{k}'^2 = 1,$$

$$O(\phi, \mathbf{e})\; \mathbf{K} = \mathbf{K}' . \tag{140}$$

One very simple solution of the problem follows immediately from the geometrical scheme (see Fig. 9)



$$\mathbf{e} = \mathbf{k} \times \mathbf{k}'$$
$$\cos\phi = \mathbf{k} \bullet \mathbf{k}'$$

Fig. 9 Additional rotation

Evidently, such a vector $\mathbf{C} = \tan\frac{\phi}{2}\,\mathbf{e}$ might be constructed by the formula

$$O(\phi, \mathbf{e})\,\mathbf{K} = \mathbf{K}' , \qquad \mathbf{C} = \tan\frac{\phi}{2}\,\mathbf{e} = \frac{\mathbf{K} \times \mathbf{K}'}{\mathbf{K} \bullet (\mathbf{K} + \mathbf{K}')} = \frac{\mathbf{k} \times \mathbf{k}'}{1 + \mathbf{k} \bullet \mathbf{k}'} . \qquad (141)$$

It is quite understandable that having done this simplest rotation after that one can rotate additionally over the axis $\mathbf{k}'$ – at this the vector $\mathbf{K}'$ itself leaves unchanged:

$$O(\varphi, \mathbf{k}')\mathbf{K}' = \mathbf{K}' .$$

12 On the form of spherical mirror in the moving reference frame (2-dimensional case)

Firstly, let us us consider 2-dimension spherical surface (circle) to which in the rest reference frame K' corresponds an equation

$$x^{'2} + y^{'2} = R^2 .$$

Through light signals, with this geometrical circle can be associated the following set of events S' in space-time:

$$S' = \left\{\ t' = \sqrt{x'^2 + y'^2}\ , x^{'2} + y^{'2} = R^2\ \right\} . \qquad (142)$$

The Lorentz transform

$$t' = ch\,\beta\,t + sh\,\beta\,x\ , \qquad x' = ch\,\beta\,x + sh\,\beta\,t\ , \qquad y' = y$$

will take this events set S' to the form

$$S : \qquad \left\{\ t = \sqrt{x^2 + y^2}\ , (ch\,\beta\,x + sh\,\beta\,t)^2 + y^2 = R^2\ \right\} . \qquad (143)$$

Excluding the time variable t, one gets for geometrical surface S in the moving reference frame K

$$(\ ch\,\beta\,x + sh\,\beta\,\sqrt{x^2 + y^2}\ )^2 + y^2 = R^2 . \qquad (144)$$

From where it follows

$$ch^2\beta\,x^2 + 2ch\,\beta\,sh\,\beta\,x\sqrt{x^2 + y^2} + sh^2\beta\,x^2 + sh^2\beta\,y^2 + y^2 = R^2 ,$$

or

$$sh\ 2\beta\ x\ \sqrt{x^2+y^2} = R^2 - ch\ 2\beta\ x^2 - ch^2\beta\ y^2\ .$$

Squaring this equation

$$sh^2 2\beta\ x^4 + sh^2 2\beta\ x^2 y^2 =$$
$$= R^4 + ch^2 2\beta\ x^4 + 2\ ch\ 2\beta\ ch^2\beta\ x^2y^2 + ch^4\beta\ y^4 - 2R^2 ch\ 2\beta\ x^2 - 2R^2 ch^2\beta\ y^2\ ,$$

one gets to

$$(x^4 + 2\ ch^2\beta\ x^2y^2 + ch^4\beta\ y^4) + R^4 - 2R^2\ (2\ sh^2\beta + 1)\ x^2 - 2R^2\ ch^2\beta\ y^2 = 0\ .$$

The equation obtained can be rewritten differently

$$(x^2 + ch^2\beta\ y^2)^2 - 2R^2(x^2 + ch^2\beta y^2) + R^4 = \ 4R^2\ sh^2\beta\ x^2\ ,$$

that is

$$[\ x^2 + ch^2\beta\ y^2 - R^2]^2 = [\ 2R\ sh\ \beta\ x\ ]^2\ .$$

So, we arrive at a second order equation

$$x^2 + ch^2\beta\ y^2 - R^2 = 2R\ sh\ \beta\ x\ \ ;$$

it easily can be simplified to the form

$$(x - 2R\ sh\ \beta)^2 + ch^2\beta\ y^2 = R^2\ ch^2\beta\ .$$

what is a canonical equation for ellipse

$$\frac{(x - 2R\ sh\ \beta)^2}{R^2\ ch^2\beta} + \frac{y^2}{R^2} = 1\ . \qquad (145)$$

13 On the form of spherical mirror in the moving reference frame, general 3-dimensional case

A spherical mirror in the rest reference frame K' is described by the equation

$$\mathbf{x}'\mathbf{x}' = R^2\ ;$$

with it can be related the special set of events S' in space-time:

$$S' = \Big\{\ t' = \sqrt{\mathbf{x}'\mathbf{x}'} = R,\ \mathbf{x}'\mathbf{x}' = R^2\ \Big\}\ . \qquad (146)$$

The Lorentz transform change it to other form

$$S: \qquad \{\ t = \sqrt{\mathbf{xx}} = R,\ [\ \mathbf{x} + \mathbf{e}\ (\ sh\ \beta\ t + (ch\ \beta - 1)\ (\mathbf{ex})\ )\ ]^2 = R^2\ \ \}\ ,$$

Excluding the time variable t, one gets an equation corresponding to a geometrical form of the same surface in the moving reference frame

$$[\ \mathbf{x} + \mathbf{e}\ (\ sh\ \beta\ \sqrt{\mathbf{xx}} + (ch\ \beta - 1)\ (\mathbf{ex})\ )\ ]^2 = R^2\ .$$

From this it follows

$$\mathbf{x}^2 + 2\ (\mathbf{ex})[\ sh\ \beta\ \sqrt{\mathbf{xx}} + (ch\ \beta - 1)\ (\mathbf{ex})\ ] + (\ sh\ \beta\ \sqrt{\mathbf{xx}} + (ch\ \beta - 1)\ (\mathbf{ex})\ )^2 = R^2\ ,$$

or

$$\mathbf{x}^2 + 2\ sh\ \beta\ (\mathbf{ex})\sqrt{\mathbf{xx}} + 2(ch\ \beta - 1)\ (\mathbf{ex})^2 +$$
$$+ sh^2\beta\ \mathbf{x}^2 + 2sh\ \beta(ch\ \beta - 1)\ (\mathbf{ex})\ \sqrt{\mathbf{xx}} + (ch\ \beta - 1)^2\ (\mathbf{ex})^2 = R^2\ ,$$

and further

$$ch^2\beta\ \mathbf{x}^2 + sh^2\beta\ (\mathbf{ex})^2 + 2sh\ \beta\ ch\ \beta\ \sqrt{\mathbf{xx}}\ (\mathbf{ex}) = R^2\ .$$

Thus, we have arrived at the equation

$$2sh\ \beta\ ch\ \beta\ \sqrt{\mathbf{xx}}\ (\mathbf{ex}) = R^2 - ch^2\beta\ \mathbf{x}^2 - sh^2\beta\ (\mathbf{ex})^2\ . \qquad (147)$$

It make sense to employ special coordinate system – such that $\mathbf{e}$ become oriented along the first axis:

$$O\mathbf{e} = (1, 0, 0), \qquad \mathbf{ex} = \mathbf{e}\tilde{O}O\mathbf{x} = (O\mathbf{e}) \bullet \mathbf{X} = X \qquad (148)$$

as a result eq. (147) will take a simpler form

$$sh\ 2\beta\ X\ \sqrt{X^2 + Y^2 + Z^2} = R^2 - ch^2\beta\ (X^2 + Y^2 + Z^2) - sh^2\beta\ X\ \ ;$$

which can be presented as

$$sh\ 2\beta\sqrt{X^2 + (Y^2 + Z^2)}\ \ X = R^2 - ch\ 2\beta\ X^2 - ch^2\beta\ (Y^2 + Z^2)\ . \qquad (149)$$

The relation obtained is readily compared with previously established eq. (144)

$$sh\ 2\beta\ x\ \sqrt{x^2 + y^2} = R^2 - ch\ 2\beta\ x^2 - ch^2\beta\ y^2$$

through the formal changes

$$x \quad \Longrightarrow \quad X\ , \qquad y^2 \quad \Longrightarrow \quad (Y^2 + Z^2)\ .$$

One may use the previously determined solution – see. (145):

$$\frac{(x - 2R\ sh\ \beta)^2}{R^2\ ch^2\beta} + \frac{y^2}{R^2} = 1\ ;$$

from where one arrives at

$$\frac{(X - 2R\ sh\ \beta)^2}{R^2\ ch^2\beta} + \frac{Y^2}{R^2} + \frac{Z^2}{R^2} = 1\ . \qquad (150)$$

Thus, for the moving observer, the spherical surface becomes an ellipsoid K.

14 The light reflection law in media

Let a light ray (incident and reflected) be propagated in a uniform media with refraction index $n > 1$. In the rest reference frame K' the reflection law preserves its form. However, there arise differences when going to a moving reference frame K.

The fact of the most significance is that when using ordinary (vacuum-based) Lorentz transformations then because the speed of light in the rest media is less than c ($kc < c$) the modulus of the light velocity does not preserve its value, as a result meaning of the formulas becomes very different.

Instead of eq. (68) now we have

$$\mathbf{a}' = \frac{\mathbf{W}'_{in}}{c} \; , \;\; \mathbf{a}'^{2} < 1 \; , \qquad \mathbf{b}' = \frac{\mathbf{W}'_{out}}{c} \; , \;\; \mathbf{b}'^{2} < 1 \; ;$$
$$\mathbf{n}' = \frac{\mathbf{W}'_{norm}}{c} \; , \qquad \mathbf{n}'^{2} < 1 \; . \tag{151}$$

General mathematical form of the reflection law in the moving reference frame (90) formally stays the same

$$\frac{\mathbf{a} \times \mathbf{n} + th\,\beta\,(\mathbf{a} - \mathbf{n}) \times \mathbf{e} + (1 - ch^{-1}\beta)\,\mathbf{e}\,[\,\mathbf{e}(\mathbf{n} \times \mathbf{a})\,]}{1 + th\,\beta\,(\mathbf{ea})} =$$
$$= \frac{\mathbf{b} \times \mathbf{n} + th\,\beta(\mathbf{b} - \mathbf{n}) \times \mathbf{e} + (1 - ch^{-1}\beta)\,\mathbf{e}\,[\,\mathbf{e}(\mathbf{n} \times \mathbf{b})\,]}{1 + th\,\beta\,(\mathbf{eb})} \; , \tag{152}$$

$$\frac{\mathbf{an}(1 - V^2) + (\mathbf{a} + \mathbf{n})\mathbf{V} + [\,V^2 + (\mathbf{aV})(\mathbf{nV})\,]}{1 + \mathbf{aV}} +$$
$$+ \frac{\mathbf{bn} + (\mathbf{b} + \mathbf{n})\mathbf{V} + [\,V^2 + (\mathbf{bV})(\mathbf{nV})\,]}{1 + \mathbf{bV}} = 0 \; , \tag{153}$$

$$\Delta = \mathbf{n}\,(\mathbf{a} \times \mathbf{b}) =$$
$$= [\,\mathbf{e}(\mathbf{n}' \times \mathbf{a}')\,]\;\frac{-2\;sh\,\beta\,(\mathbf{a}'\mathbf{n}')}{(ch\,\beta - sh\,\beta\,\mathbf{en}')\,(ch\,\beta - sh\,\beta\,\mathbf{ea}')\,(ch\,\beta - sh\,\beta\,\mathbf{eb}')} \; , \tag{154}$$

however, one must take into account that the lengths of the vectors involved $\mathbf{a}, \mathbf{b}, \mathbf{n}$ are different from 1.

Let us calculate the length of the light vector in the moving reference frame. We should start with the transformation law

$$\mathbf{W} = \frac{\mathbf{W}' + \mathbf{e}\,[\,-sh\,\beta + (ch\,\beta - 1)\,(\mathbf{e}\,\mathbf{W}')\,]}{ch\,\beta - sh\,\beta\,\mathbf{e}\,\mathbf{W}'} \; . \tag{155}$$

In the rest reference frame K' the speed of light is k (it is convenient to divide all velocities by c):

$$\mathbf{W}' = k\mathbf{w}' \; , \qquad \mathbf{w}'^{2} = 1 \; .$$

Eq. (155) reads as

$$\mathbf{W} = \frac{k\mathbf{w}' + \mathbf{e}\,[\,-sh\,\beta + k(ch\,\beta - 1)\;\mathbf{ew}'\,]}{ch\,\beta - k\;sh\,\beta\;\mathbf{ew}'} \; . \tag{156}$$

The modulus of $\mathbf{W}$ is

$$\mathbf{W}^2 = \left[\frac{k\mathbf{w}' + \mathbf{e}\,[\,-sh\,\beta + k(ch\,\beta - 1)\,\mathbf{e}\mathbf{w}'\,]}{ch\,\beta - k\,sh\,\beta\,\mathbf{e}\mathbf{w}'}\right]^2 =$$

$$= \frac{k^2 + 2k(\mathbf{w}'\mathbf{e})[\,-sh\,\beta + k(ch\,\beta - 1)\,\mathbf{e}\mathbf{w}'\,] + [\,-sh\,\beta + k(ch\,\beta - 1)\,\mathbf{e}\mathbf{w}'\,]^2}{[ch\,\beta - k\,sh\,\beta\,\mathbf{e}\mathbf{w}']^2} .$$

With the notation

$$\mathbf{w}'\mathbf{e} = \mu\,, \qquad \mu \in [-1, +1] \tag{157}$$

one gets to

$$\mathbf{W}^2 = \frac{k^2 + 2k\mu[\,-sh\,\beta + k\mu(ch\,\beta - 1)] + [\,-sh\,\beta + k\mu(ch\,\beta - 1)\,]^2}{(ch\,\beta - k\mu\,sh\,\beta)^2} .$$

With simple calculation

$$\begin{aligned}
k^2 + 2k\mu[\,-sh\,\beta + k\mu(ch\,\beta - 1)] + [\,-sh\,\beta + k\mu(ch\,\beta - 1)\,]^2 =\\
= k^2 - 2k\mu\,sh\,\beta + 2k^2\mu^2(ch\,\beta - 1) +\\
+sh^2\beta - 2k\mu\,sh\,\beta(ch\,\beta - 1) + k^2\mu^2(ch^2\beta - 2ch\,\beta + 1) =\\
= k^2 - 2k\mu\,sh\,\beta\,ch\,\beta + sh^2\beta - k^2\mu^2 + k^2\mu^2 ch^2\beta =\\
= (k^2 - 1) + (ch^2\beta - 2k\mu\,sh\,\beta\,ch\,\beta + k^2\mu^2 sh^2\beta) =\\
= (k^2 - 1) + (ch\beta - k\mu\,sh\,\beta\,)^2
\end{aligned}$$

we arrive at

$$\mathbf{W}^2 = 1 - \frac{(1 - k^2)}{(ch\,\beta - k\mu\,sh\,\beta)^2} \;<\; 1 . \tag{158}$$

Thus, the speed of light is not invariant quantity under the ordinary Lorentz transformations; it depends on mutual orientation of $\mathbf{w}'$ *and* $\mathbf{e}$ *– remembering* $\mu = \mathbf{e}\mathbf{w}'$. *In any moving reference frame* K, *the light velocity* W *in the media is less than vacuum velocity.*

Eq. (158) can be written as

$$\mathbf{W}^2 = 1 - (1 - k^2)\,\frac{(1 - V^2)}{(1 - \mu k V)^2} . \tag{159}$$

There are two simple cases of (anti) parallel motion of the light and reference frame:

$$\begin{aligned}
\mu = +1 : \qquad W = \frac{V + k}{1 - kV}\,,\\
\mu = -1 : \qquad W = \frac{V - k}{1 + kV}\,;
\end{aligned} \tag{160}$$

in usual units these look as

$$\begin{aligned}
\mu = +1 : \qquad W = \frac{V + W'}{1 - W'V/c^2}\,,\\
\mu = -1 : \qquad W = \frac{V - W'}{1 + W'V/c^2}\,.
\end{aligned} \tag{161}$$

From the formulas obtained (158) and (155), one can readily derive relationship describing aberration of the light in the moving reference frame. Indeed,

$$\mathbf{W} = \frac{k\mathbf{w}' + \mathbf{e}\,[\,-sh\,\beta + k\mu(ch\,\beta - 1)\,]}{ch\,\beta - k\mu\; sh\,\beta} = W\;\mathbf{w}\;,$$

$$W = \sqrt{1 - \frac{(1-k^2)}{(ch\,\beta - k\mu\; sh\,\beta)^2}}\;, \qquad \mu = \mathbf{e}\mathbf{w}'\;. \tag{162}$$

So that a unit vector of the light in the K frame is given by

$$\mathbf{w} = \left[1 - \frac{(1-k^2)}{(ch\,\beta - k\mu\; sh\,\beta)^2}\right]^{-1/2} \frac{k\mathbf{w}' + \mathbf{e}\,[\,-sh\,\beta + k\mu(ch\,\beta - 1)\,]}{ch\,\beta - k\mu\; sh\,\beta} =$$

$$= \frac{k\mathbf{w}' + \mathbf{e}\,[\,-sh\,\beta + k\mu(ch\,\beta - 1)\,]}{\sqrt{(ch\,\beta - k\mu\; sh\,\beta)^2 - 1 + k^2}}\;. \tag{163}$$

This relation is much simplified when the vectors $\mathbf{w}'$ and $\mathbf{e}$ are perpendicular to each other:

$$\mu = 0, \qquad \mathbf{w} = \frac{k\mathbf{w}' - sh\,\beta\;\mathbf{e}}{\sqrt{sh^2\beta + k^2}}\;, \tag{164}$$

therefore the angle of aberration is as follows:

$$\tan\alpha = \frac{sh\,\beta}{k}\;, \tag{165}$$

or in ordinary units

$$\tan\alpha = \frac{V}{kc}\;\frac{1}{\sqrt{1 - V^2/c^2}}\;.$$

It should be especially noted one other aspect of the problem: eq. (162) means that the light velocity in the reference frame K is a function of direction of the propagation of the light. This fact is of most significance because it change basically the general structure of special relativity in presence of a media. In such circumstances there appears an absolute reference frame related to the rest media, the reference frame K'. In the reference frame K', the light velocity is an isotropic quantity that preserves its value in all space directions. In any other reference frame, moving K, the light velocity is anisotropic – it is a function of directions.

It is convenient to write eq. (158) in the form

$$1 - \mathbf{W}^2 = (1 - W'^2)\;\;\frac{(1 - V^2)}{(1 - \mathbf{W}'\mathbf{V})^2}\;. \tag{166}$$

Here, by symmetry reasons, we can restrict ourselves to one angle variable so that

$$1 - W^2(\alpha) = (1 - W'^2)\;\;\frac{(1 - V^2)}{(1 - VW'\;\cos\alpha)^2}\;. \tag{167}$$

We may reach the formal simplicity with the use of trigonometrical parametrization:

$$V = th\ \beta\ , \qquad W' = th\ B'\ , \qquad W(\alpha) = th\ B(\alpha)\ ,$$

them eq. (167) gives

$$ch\ B(\alpha) = ch\ B'\ ch\ \beta - sh\ B'\ sh\ \beta\ \cos\alpha\ . \tag{168}$$

15 The light and the tensor formalism of 4-velocities u^a

In general, the 4-velocity vector is defined as [30]

$$u^a = (u^0, \mathbf{u}) = (\ \frac{1}{\sqrt{1-W^2}}\ ,\ \frac{\mathbf{W}}{\sqrt{1-W^2}}\ )\ , \qquad \mathbf{W} = W\ \mathbf{w}\ . \tag{169}$$

To have the case of the light one should set $W = 1$, then eq. (169) gives

$$u^a = (u^0, \mathbf{u}) = \infty\ (\ 1,\ \mathbf{w}\ )\ . \tag{170}$$

To add some details at this limiting procedure it is convenient to employ the following parametrization

$$\mathbf{W} = th\ B\ \mathbf{w}\ , \qquad u^a = (\ ch\ B,\ sh\ B\ \mathbf{w}\ )\ ,\ \ B \in [0, +\infty)\ ,$$

to the light limit there corresponds $B \longrightarrow +\infty$:

$$\begin{gathered}
\lim_{B\to\infty} ch\ B = \lim_{B\to\infty} \frac{e^B + e^{-B}}{2} = +\infty, \\
\lim_{B\to\infty} sh\ B = \lim_{B\to\infty} \frac{e^B - e^{-B}}{2} = +\infty. \\
\lim_{B\to\infty} th\ B = 1\ , \qquad u^a = \infty\ (\ 1,\ \mathbf{w}\ )
\end{gathered}$$

and

$$\begin{gathered}
\frac{1}{\sqrt{1-W^2}} = ch\ B = \frac{e^B + e^{-B}}{2}, \qquad ch^2 B \approx \frac{e^{2B}+2}{4} \sim \frac{\infty^2}{4} + \frac{1}{2}\ , \\
\frac{W}{\sqrt{1-W^2}} = sh\ B = \frac{e^B - e^{-B}}{2}, \qquad sh^2 B \approx \frac{e^{2B}-2}{4} \sim \frac{\infty^2}{4} - \frac{1}{2} \\
ch^2 B - sh^2 B \approx 1\ , \\
W^2 = \frac{e^{2B}-2}{e^{2B}+2} = 1 - \frac{4}{e^{2B}+2} \sim \frac{\infty^2 - 2}{\infty^2 + 2} = 1 - \frac{4}{\infty^2+2}\ , \\
ch\ B = \frac{\sqrt{e^{2B}+2}}{2} = \frac{e^B}{2}\ \sqrt{1+2e^{-2B}} = \frac{e^B}{2}\ (1 + e^{-2B}) \sim \frac{\infty}{2}\ (1+\infty^{-2})\ , \\
sh\ B = \frac{\sqrt{e^{2B}-2}}{2} = \frac{e^B}{2}\ \sqrt{1-2e^{-2B}} = \frac{e^B}{2}\ (1 - e^{-2B}) \sim \frac{\infty}{2}\ (1-\infty^{-2})\ .
\end{gathered} \tag{171}$$

Now let us return to eq. (169) at very big B-s:

$$u^a = (\frac{e^B}{2} (1+e^{-2B}),\ \frac{e^B}{2} (1-e^{-2B}) \, \mathbf{w} \ ) \sim \infty \, (1, \mathbf{w}) \ ; \tag{172}$$

and examine how will act the Lorentz transform on such a limiting 4-vector First, consider simple 1-dimensional case $\mathbf{W} = (W, 0, 0)$:

$$u^a = (\frac{e^B}{2} (1+e^{-2B}),\ \ \frac{e^B}{2} (1-e^{-2B}), 0, 0 \ ) \sim \infty \, (1,1,0,0) \ .$$

The Lorentz transformation

$$t' = ch\,\beta\, t + sh\,\beta\, x \ , \qquad x' = \ sh\,\beta\, t + ch\,\beta\, x$$

act as follows

$$u'^0 = ch\,\beta\ chB + sh\,\beta\ sh\,B = ch\,(\beta + B) \ ,$$
$$u'^1 = sh\,\beta\ ch\,B + ch\,\beta\ sh\,B = sh\,(\beta + B) \ \ ,$$

that is

$$u'^a = (e^{\beta+B}\ \frac{1+e^{-2(\beta+B)}}{2}\ , e^{\beta+B}\ \frac{1-e^{-2(\beta+B)}}{2},\ 0,\ 0) \sim \infty'(1,1,0,0) \tag{173}$$

where the notation is used

$$\infty' = \frac{e^{\beta+B}}{2} \ .$$

Here we should see the parameter β as a finite one whereas the B must be seen as infinity.

Generalization of the above analysis to the case of arbitrary Lorentz transformation can be done quite easily. Here we have

$$u'^a = \frac{1}{\sqrt{1-W'^2}} \ (1,\ \mathbf{W}' \ ) \ ,$$

where

$$W' = \sqrt{1 - \frac{(1-W^2)}{(ch\,\beta + W(\mathbf{ew})\ sh\,\beta)^2}} \ , \qquad \mathbf{W}' = W' \ \mathbf{w}' \ ,$$
$$\mathbf{w}' = \frac{\mathbf{W} + \mathbf{e}\ [\ sh\,\beta + W(\mathbf{ew})(ch\,\beta - 1)\ ]}{\sqrt{(ch\,\beta + W(\mathbf{ew})\ sh\,\beta)^2 - (1-W^2)}} \ ,$$

so

$$\mathbf{W}' = \frac{ch\,\beta + W(\mathbf{ew})\ sh\,\beta}{\sqrt{1-W^2}} \ \ (\ 1 \ , \ \frac{\mathbf{W} + \mathbf{e}\ [\ sh\,\beta + W(\mathbf{ew})(ch\,\beta - 1)\ ]}{ch\,\beta + W(\mathbf{ew})\ sh\,\beta} \ ) \ . \tag{174}$$

From this, for the light we will produce

$$W \longrightarrow \ 1, \qquad \frac{1}{\sqrt{1-W^2}} \longrightarrow \frac{e^B}{2}\ \sqrt{1+2e^{-2B}} \ \ \longrightarrow \ \ \infty \ , \qquad \mathbf{W}' \longrightarrow \mathbf{w}',$$

and previous relation takes the form

$$u^{'a} = \infty \, \varphi' \, (\, 1 \, , \, \mathbf{w}') \tag{175}$$

where the function φ' at the ∞ symbol is

$$\varphi'(\beta, \mathbf{e}, \mathbf{w}) \;\equiv\; [ch \; \beta + (\mathbf{ew}) \; sh \; \beta] = \frac{1 + \mathbf{wV}}{\sqrt{1 - V^2}} \; , \tag{176}$$

and the new light vector $\mathbf{w}'$ is given by

$$\mathbf{w}' = \frac{\mathbf{w} + \mathbf{e} \; [\; sh \; \beta + (\mathbf{ew})(ch \; \beta - 1) \;]}{ch \; \beta + (\mathbf{ew}) \; sh \; \beta} \; . \tag{177}$$

Thus, action of the Lorentz transform on light 4-velocity vector can be summarized in symbolical form as follows

$$\begin{gathered} L \; [\; u^a = \infty \; (1, \mathbf{w}) \;] = \varphi' \; \infty \; (1, \mathbf{w}') \; , \\ \varphi' = ch \; \beta + (\mathbf{ew}) \; sh \; \beta = \frac{1 + \mathbf{wV}}{\sqrt{1 - V^2}} \; . \\ \mathbf{w}' = \frac{\mathbf{w} + \mathbf{e} \; [\; sh \; \beta + (\mathbf{ew})(ch \; \beta - 1) \;]}{ch \; \beta + (\mathbf{ew}) \; sh \; \beta} \; , \end{gathered} \tag{178}$$

and for usual (non light like) velocities

$$\begin{gathered} L \; [\; u^a \;] = u^{'a} \; , \\ u^a = \frac{1}{\sqrt{1 - W^2}} \; (\; 1, \mathbf{W} \;) \; , \qquad u^{'a} = \frac{1}{\sqrt{1 - W'^2}} \; (\; 1, \mathbf{W}' \;) = \\ = \frac{1}{\sqrt{1 - W^2}} \; \frac{(1 + \mathbf{WV})}{\sqrt{1 - V^2}} \; \Big(\; 1 \; , \; \frac{\mathbf{W} + \mathbf{e} \; [\; sh \; \beta + (ch \; \beta - 1)\mathbf{eW} \;]}{ch \; \beta + W(\mathbf{ew}) \; sh \; \beta} \; \Big) \; . \end{gathered} \tag{179}$$

Evidently, eq. (179) will coincide with (178) at the limit $W \to 1$.

16 Relativistic velocities and Lobachewski geometry

The problem considered of describing the light 4-velocities in contrast to usual 4-velocities has an interesting geometrical interpretation[8] . Indeed, every 4-vector $u^a = (u^0, \mathbf{u})$ obeys the condition

$$(u^0)^2 - \mathbf{u}^2 = 1 \tag{180}$$

which gives the known realization of the 3-dimensional Lobachewski space H_3 of constant negative curvature as a surface in a pseudo Euclidean 4-space. There exist one-to-one correspondence between 4-velocities and points of the geometrical space of negative constant curvature H_3.

[8] The composition law for velocities is intimately related with hyperbolic geometry (i.e. geometry on spaces with constant negative curvature), as was first pointed out by A. Sommerfeld [87], V. Varičak , Alfred A. Robb [88-90], and Émile Borel [91]. More recently the subject was elaborated on by Abraham A. Ungar (Ungar [92]; many others aspects see in [87].

One can see that all points at spatial infinity making the bounding domain $\bar{H}_3$ are described by

$$\underline{\bar{H}_3 :} \qquad u^0 = \frac{e^B}{2}(1 + e^{-2B}) \; , \mathbf{u} = \frac{e^B}{2}(1 - e^{-2B})\mathbf{w} \; , \mathbf{w}^2 = 1 \; , \quad B \longrightarrow \infty \tag{181}$$

and just such (infinite bound) points should be associated with all light 4-velocities.

In the parametrization (B, θ, ϕ)

$$\begin{gathered} u^0 = ch\ B \; , \qquad \mathbf{u} = sh\ B\ \mathbf{w}, \\ \mathbf{w} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \; , \\ B \in [0, +\infty) \; , \qquad \theta \in [0, \pi], \qquad \phi \in [0, 2\pi] \end{gathered} \tag{182}$$

the infinite bound $\bar{H}_3$ is defined by

$$\underline{\bar{H}_3 :} \qquad B = \infty, \qquad \theta \in [0, \pi], \qquad \phi \in [0, 2\pi] \; . \tag{183}$$

With the help of new variables $(W = th\ B, \theta, \phi)$

$$\mathbf{W} = \frac{\mathbf{u}}{u_0} = (th\ B)\ \mathbf{W} = W\ \mathbf{w} \; , \qquad W = th\ B \ \in [0, +1) \; .$$

this bound set $\bar{H}_3$ can be described in terms of finite quantities:

$$\underline{\bar{H}_3 :} \qquad W = 1, \qquad \theta \in [0, \pi], \qquad \phi \in [0, 2\pi] \; . \tag{184}$$

Geometrical considerations can be helpful additional instrument in studying the problem. For instance, let us pose the question: to what extent will differ ordinary velocity and light one. As an appropriate characteristic let us try the geometrical distance between respective points in the space H_3. It is known the formula for metric of H_3 in coordinates (B, θ, ϕ) :

$$dl^2 = dB^2 + sh\ ^2B(d\theta^2 + \sin^2\theta d\phi^2) \; , \qquad (W = th\ B) \; . \tag{185}$$

The distance between two points, (W_1, θ_0, ϕ_0) and $(W_2 = 1, \theta_0, \phi_0)$, along the fixed direction (θ_0, ϕ_0) is given by

$$\begin{aligned} l = \int_{B_1}^{B_2} \sqrt{dB^2 + sh\ ^2B(d\theta^2 + \sin^2\theta d\phi^2)} \Big|_{\theta_0, \phi_0} = \\ = \int_{B_1}^{B_2} dB = B_2 - B_1 = \infty - \ arcth\ W_1 \; . \end{aligned} \tag{186}$$

Thus, in geometrical terms, any ordinary velocity W_1 being just slightly different from the light one $W_2 = 1$ is located the infinite distance from the light velocity: $l = \infty - B_1$. It may be emphasized geometrical sense of W: it provides us with minimal geometrical distance in the corresponding Lobachewsky space:

$$l = arcth\, W \;, \qquad W = th\, l \;; \tag{187}$$

in ordinary units it looks as

$$\frac{W}{c} = th\, \frac{l}{R} \;, \tag{188}$$

where R stands for the curvature radius of the space H_3. (all such spaces, $\{ \; R, H_3 \; \}$ are similar to each other.).

One other distance characteristic may be defined in the case of differently oriented velocities (consider specially simple example):

$$\begin{gathered} \mathbf{W}_1 = (W \cos \phi_1, W \sin \phi_1, 0) \;, \\ \mathbf{W}_1 = (W \cos \phi_2, W \sin \phi_2, 0), \\ L = \{ \; (W, \theta = \pi/2, \phi \in [\phi_1, \phi_2 \;) \; \} \;; \end{gathered} \tag{189}$$

that is

$$\begin{aligned} L = \int_{\phi_1}^{\phi_2} \sqrt{dB^2 + sh\;^2 B (d\theta^2 + \sin^2\theta d\phi^2)} \Big|_{W, \theta = \pi/2} = \\ = sh\; B\; (\phi_2 - \phi_1) = \frac{W}{\sqrt{1 - W^2}}\; (\phi_2 - \phi_1) \;. \end{aligned} \tag{190}$$

In geometrical terms, the Lorentz transforms of the velocity vectors

$$\begin{gathered} \mathbf{W}' = \frac{\mathbf{W} + \mathbf{e}\; [\; sh\; \beta + (ch\; \beta - 1)\; \mathbf{e}\; \mathbf{W}\;]}{1 + \mathbf{V}\; \mathbf{W}}\; \sqrt{1 - V^2} = \\ = \; \frac{\mathbf{V} + \mathbf{e}(\mathbf{e}\mathbf{W})}{1 \; + \; \mathbf{V}\; \mathbf{W}} + \frac{\mathbf{W} - \mathbf{e}(\mathbf{e}\; \mathbf{W})}{1 + \mathbf{V}\; \mathbf{W}}\; \sqrt{1 - V^2} \;, \\ \frac{1}{\sqrt{1 - W'^2}} = \; \frac{1}{\sqrt{1 - W^2}}\; \frac{(1 + \mathbf{W}\mathbf{V})}{\sqrt{1 - V^2}} \end{gathered} \tag{191}$$

can be interpreted as special group actions on the points of Lobachewsky space H_3. To the case of the light in vacuum there corresponds such group actions upon the points of the bound $\bar{H}_3$ of the space (located in the infinity); at this the 3-dimensional problem (16.11) effectively reduces to 2-dimensional one because the normalization condition holds

$$\mathbf{W}'^2 = \mathbf{W}^2 = 1 \;.$$

However, for the light in the media both vectors $\mathbf{W}'$ and $\mathbf{W}$ do not belong this bound $\bar{H}_3$, they both are ordinary finite points in the space H_3.

17 On relativistic transformation of the geometrical form of the surface to a moving reference frame in presence of a media

(IN VACUUM)

Let us recall the above method to determine the form of any rigid surface in a moving reference frame Let the surface S' be given in the rest reference frame

$$S' : \qquad \{ \varphi(\mathbf{x}) = 0 \} \; . \tag{192}$$

To this geometrical structure one can pose in correspondence a special set of events in space-time:

$$\{ (t', \mathbf{x}') : \qquad \mathbf{x}' = \mathbf{W}'t' \; , \;\; \varphi(\mathbf{x}') = 0 \, \}$$

which with respect to Lorentz formulas

$$t' = ch\, \beta \, t + sh\, \beta \, \mathbf{e} \, \mathbf{x} \; ,$$
$$\mathbf{x}' = \mathbf{e} \, sh\, \beta \, t + \mathbf{x} + (ch\, \beta - 1) \, \mathbf{e} \, (\mathbf{e}\mathbf{x})$$

will take other form

$$\{ (t, \mathbf{x}) : \qquad \mathbf{x} = \mathbf{W}t \; , \; \varphi \, [\, \mathbf{x} + \mathbf{e} \, (\, sh\, \beta \, t + (ch\, \beta - 1) \, \mathbf{e}\mathbf{x} \,) \,] = 0 \, \} \; .$$

In order to obtain an equation for the transformed geometrical surface S, one should exclude the time variable t:

$$\mathbf{x} = \mathbf{W}t, \; t = \sqrt{\mathbf{x}^2} \; , \qquad (\mathbf{W}^2 = 1 \,) \; ; \tag{193}$$

so one produces

$$S : \qquad \varphi \, [\, \mathbf{x} + \mathbf{e} \, (\, sh\, \beta \, \sqrt{\mathbf{x}^2} + (ch\, \beta - 1) \, \mathbf{e}\mathbf{x} \,) \,] = 0 \; . \tag{194}$$

(IN THE MEDIA)

In the same line one can act for the case of a uniform media when the light velocity $\mathbf{W}'^2$ changes in accordance with special law:

$$\mathbf{W}^2 = 1 - (1 - W'^2) \, \frac{(1 - V^2)}{(1 - VW' \cos \alpha)^2} \; . \tag{195}$$

Again, one starts with certain S' surface in the rest reference frame

$$S' : \qquad \{ \varphi(\mathbf{x}) = 0 \}$$

With the help of light signals, one to the S' structure can be referred a set of events in space-time:

$$\{ (t', \mathbf{x}') : \qquad \mathbf{x}' = \mathbf{W}'t' \; , \; \sqrt{\mathbf{x}'^2} = kt' \; , \; \varphi(\mathbf{x}') = 0 \, \}.$$

Under Lorentz formulas

$$t' = ch\, \beta \, t + sh\, \beta \, \mathbf{e} \, \mathbf{x} \; ,$$
$$\mathbf{x}' = \mathbf{x} + \mathbf{e} \, (\, sh\, \beta \, t + (ch\, \beta - 1) \, \mathbf{e}\mathbf{x} \,) \; .$$

these events change into

$$\{ (t, \mathbf{x}) : \qquad \mathbf{x} = \mathbf{W}t \; , \; \varphi \, [\mathbf{x} + \mathbf{e} \, (\, sh\, \beta \, t + (ch\, \beta - 1) \, \mathbf{e}\mathbf{x} \,) \,] = 0 \, \} \; .$$

Now, it is the point to exclude the time variable t , however one must take into account the rule (195):

$$\mathbf{x} = \mathbf{W}t, \qquad t = \frac{\sqrt{\mathbf{x}^2}}{\sqrt{\mathbf{W}^2}} \, ,$$

so we arrive at

$$S : \qquad \varphi \left[\mathbf{x} + \mathbf{e} \, (\, sh\, \beta \, \frac{\sqrt{\mathbf{x}^2}}{\sqrt{\mathbf{W}^2}} \, + (ch\, \beta - 1) \, (\mathbf{ex}) \, \right] = 0 \, , \tag{196}$$

where

$$\mathbf{W}^2 = 1 - (1 - W^{'2}) \, \frac{(1 - V^2)}{(1 - VW' \cos \alpha)^2} \, .$$

The relation (196) describing the change in the form of a rigid surface in presence of media gives a general solution to the problem under consideration. However, that solution in not explicit and in any particular case we need some additional analysis to have in hand the new transformed form of the surface in fact.

18 Modified Lorenz transforms in a uniform media

Formal scheme of Special Relativity in a uniform media can be constructed on the base of the light velocity in the media kc ([94-96]; see the recent paper [97]):

$$c \qquad \Longrightarrow \qquad k\, c, \qquad k < 1 \, .$$

Modified Lorentz formulas are defined so that the new interval preserves its form:

$$k^2 c^2 \, t^2 - \mathbf{x}^2 = k^2 c^2 \, t^{'2} - \mathbf{x}^{'2} \, . \tag{197}$$

In the simplest 1-dimensional case

$$k^2 c^2 \, t^2 - x^2 = k^2 c^2 \, t^{'2} - x^{'2}$$

new Lorentz formulas look as

$$x' = \frac{x - Vt}{\sqrt{1 - V^2/k^2c^2}}, \qquad t' = \frac{t - xV/k^2c^2}{\sqrt{1 - V^2/k^2c^2}} \, ;$$

indeed, it is easily verified identity

$$k^2 c^2 \, t^{'2} - \mathbf{x}^{'2} = \frac{1}{1 - V^2/k^2c^2} \, \left[k^2c^2 (t - \frac{xV}{k^2c^2})^2 - (x - Vt)^2 \right] = k^2 c^2 \, t^2 - \mathbf{x}^2 \, .$$

From (198) it follows the invariance of the light velocity $W_{light} = kc$ in the media:

$$k^2 c^2 \, t^2 - x^2 = 0, \qquad k^2 c^2 \, t^{'2} - x^{'2} = 0 \; : \qquad \Longrightarrow \qquad \frac{x^2}{t^2} = \frac{x^{'2}}{t^{'2}} = k^2 c^2 = \text{inv} \, .$$

The modified Lorentz formulas can be written

$$x' = \frac{x - \frac{V}{kc} kct}{\sqrt{1 - V^2/k^2c^2}}, \qquad kct' = \frac{kct - x\,\frac{V}{kc}}{\sqrt{1 - V^2/k^2c^2}}\;; \tag{198}$$

that is in the new variables

$$kct \Longrightarrow t, \qquad x \Longrightarrow x, \qquad \frac{V}{kc} \Longrightarrow V$$

they look

$$x' = \frac{x - Vt}{\sqrt{1 - V^2}}, \qquad t' = \frac{t - x\,V}{\sqrt{1 - V^2}}\;.$$

Extension to Lorentz transforms with arbitrary velocity vector $\mathbf{V}$ is given evidently as follows

$$\begin{aligned} t' &= ch\;\beta\;t + sh\;\beta\;\mathbf{e}\,\mathbf{x}\;, \\ \mathbf{x}' &= \mathbf{e}\;sh\;\beta\;t + \mathbf{x} + (ch\;\beta - 1)\;\mathbf{e}\;(\mathbf{ex}) = \\ &= [\;\mathbf{x} - \mathbf{e}\;(\mathbf{ex})\;] + \mathbf{e}\;[sh\;\beta\;t + ch\;\beta\;(\mathbf{ex})\;] \end{aligned} \tag{199}$$

where (kct) stands for t and

$$\begin{aligned} &\frac{\mathbf{V}}{kc} = \mathbf{e}\;th\;\beta\;, \qquad \mathbf{e}^2 = 1\;, \\ \frac{1}{\sqrt{1 - V^2/k^2c^2}} = ch\;\beta, \qquad &\frac{V/kc}{\sqrt{1 - V^2/k^2c^2}} = sh\;\beta\;. \end{aligned} \tag{200}$$

In ordinary variables, eq. (199) has the form

$$\begin{aligned} t' &= \frac{t + \mathbf{Vx}/k^2c^2}{\sqrt{1 - V^2/k^2c^2}}\;, \\ \mathbf{x}' &= [\;\mathbf{x} - \mathbf{e}\;(\mathbf{ex})\;] + \frac{\mathbf{e}(\mathbf{ex}) + \mathbf{V}t}{\sqrt{1 - V^2/k^2c^2}}\;. \end{aligned} \tag{201}$$

From (201) one can readily produce the modified light velocity addition rule

$$\begin{aligned} \mathbf{W}' &= \frac{[\;\mathbf{W} - \mathbf{e}\;(\mathbf{eW})\;] + \mathbf{e}\;[sh\;\beta + ch\;\beta\;(\mathbf{eW})\;]}{ch\;\beta + sh\;\beta\;\mathbf{e}\;\mathbf{W}} = \\ &= \frac{\mathbf{W} - \mathbf{e}\;(\mathbf{eW})}{1 + th\;\beta\;\mathbf{e}\;\mathbf{W}}\;\; ch^{-1}\;\beta \;\;+\;\; \frac{\mathbf{e}\;th\;\beta + \mathbf{e}(\mathbf{eW})}{1 + th\;\beta\;\mathbf{e}\;\mathbf{W}} \end{aligned} \tag{202}$$

or in ordinary units

$$\mathbf{W}' = \frac{\mathbf{W} - \mathbf{e}\;(\mathbf{eW})}{1 + \mathbf{V}\;\mathbf{W}/k^2c^2}\;\;\sqrt{1 - V^2/k^2c^2}\;\;+\;\;\frac{\mathbf{V}\;+\mathbf{e}(\mathbf{eW})}{1 + \mathbf{V}\;\mathbf{W}/k^2c^2}\;.$$

It may be verified straightforwardly that the value of light velocity in the media is invariant under modified Lorentz formula (202):

$$\mathbf{W}^2 = k^2c^2; \qquad \Longrightarrow \qquad \mathbf{W}'^2 = k^2c^2 \tag{203}$$

Indeed, starting from

$$\mathbf{W}'/kc = \left[\frac{\mathbf{W}}{kc} + \mathbf{e} \, [\, \frac{V}{kc\, \sqrt{1-V^2/k^2c^2}} + (\frac{1}{\sqrt{1-V^2/k^2c^2}} - 1) \, \frac{\mathbf{eW}}{kc} \,) \right] \times$$

$$\times \left[\frac{1}{\sqrt{1-V^2/k^2c^2}} + \frac{V\,(\mathbf{eW})}{k^2c^2\, \sqrt{1-V^2/k^2c^2}} \right]^{-1}$$

for the first factor we have

$$\left[\frac{\mathbf{W}}{kc} + \mathbf{e} \, [\, \frac{V}{kc\, \sqrt{1-V^2/k^2c^2}} + (\frac{1}{\sqrt{1-V^2/k^2c^2}} - 1) \, \frac{\mathbf{eW}}{kc} \,) \right]^2 =$$

$$= \frac{\mathbf{W}^2}{k^2c^2} + 2\, \frac{(\mathbf{eW})}{k^2c^2} \left[\frac{V}{\sqrt{1-V^2/k^2c^2}} + (\, \frac{1}{\sqrt{1-V^2/k^2c^2}} - 1) \, (\mathbf{eW}) \right] +$$

$$+ \frac{V^2}{k^2c^2(1-V^2/k^2c^2)} + 2\, \frac{V\,(\mathbf{eW})}{k^2c^2\sqrt{1-V^2/k^2c^2}} \, (\frac{1}{\sqrt{1-V^2/k^2c^2}} - 1) +$$

$$+ \frac{(\mathbf{eW})^2}{k^2c^2} \left(\frac{1}{1-V^2/k^2c^2} - \frac{2}{\sqrt{1-V^2/k^2c^2}} + 1 \right) =$$

$$= (\frac{\mathbf{W}^2}{k^2c^2} - 1) \; + \; 1 + \frac{V^2/k^2c^2}{1-V^2/k^2c^2} + \frac{2V\,(\mathbf{eW})}{kc\,(1-V^2/k^2c^2)} + \frac{(\mathbf{eW})^2}{k^2c^2} \, [\frac{1}{1-V^2/k^2c^2} - 1] =$$

$$= (\frac{\mathbf{W}^2}{k^2c^2} - 1) + \left[\frac{1}{\sqrt{1-V^2/k^2c^2}} + \frac{V\,(\mathbf{eW})}{k^2c^2\sqrt{1-V^2/k^2c^2}} \right]^2 .$$

Therefore, we have arrived at the rule

$$\mathbf{W}'^2 = k^2c^2 + (\mathbf{W}^2 - k^2c^2) \left[\frac{1}{\sqrt{1-V^2/k^2c^2}} + \frac{V\,(\mathbf{eW})}{k^2c^2\sqrt{1-V^2/k^2c^2}} \right]^{-2} , \tag{204}$$

it can be rewritten as

$$\mathbf{W}'^2 = k^2c^2 + \frac{\mathbf{W}^2 - k^2c^2}{[ch\;\beta + sh\;\beta\;\mathbf{eW}/kc]^2} \tag{205}$$

For the case of the light, eq. (205) reduces to eq. (203)

19 On transforming the form of rigid surface when going to a moving reference frame in presence of a uniform media, with modified Lorentz formulas

(IN MEDIA)

General method remains the same. However, now the modified Lorentz formulas should be used. Let in the rest reference frame K' certain surface be given

$$S' : \qquad \{ \, \varphi(\mathbf{x}) = 0 \, \} \, .$$

With the surface S' one may associate (with the help of light signal in the media) quite definite set of space-time events:

$$\{ \ (t', \mathbf{x}') : \qquad \mathbf{x}' = \mathbf{W}' t' \ , \ \ \varphi(\mathbf{x}') = 0 \ , \ W = kc \ \}.$$

With the help of modified Lorentz formulas

$$(kct') = (kct) \ ch \ \beta + sh \ \beta \ \mathbf{e} \ \mathbf{x} \ ,$$
$$\mathbf{x}' = \mathbf{e} \ sh \ \beta \ (kct) + \mathbf{x} + (ch \ \beta - 1) \ \mathbf{e} \ (\mathbf{e}\mathbf{x}) \tag{206}$$

where

$$ch \ \beta = \frac{1}{\sqrt{1 - V^2/k^2 c^2}}, \qquad sh \ \beta = \frac{V}{kc \ \sqrt{1 - V^2/k^2 c^2}}$$

the set of events changes to

$$\{ \ (t, \mathbf{x}) : \qquad \mathbf{x} = \mathbf{W} t \ , \ \varphi \ [\ \mathbf{x} + \mathbf{e} \ (\ kct \ sh \ \beta + (ch \ \beta - 1) \ \mathbf{e}\mathbf{x} \ ) \ ] = 0 \ \} \ . \tag{207}$$

In order to have an equation for the surface S in the reference frame K, one should exclude the time variable t:

$$\mathbf{x} = \mathbf{W} t, \qquad kct = \sqrt{\mathbf{x}^2} \ ;$$

so that

$$S : \qquad \varphi \ [\ \mathbf{x} + \mathbf{e} \ (\ sh \ \beta \ \sqrt{\mathbf{x}^2} + (ch \ \beta - 1) \ \mathbf{e}\mathbf{x} \ ) \ ] = 0 \ . \tag{208}$$

In eq. (208), the media's presence enters through the modified hyperbolic functions:

$$ch \ \beta = \frac{1}{\sqrt{1 - V^2/k^2 c^2}} \ , \qquad sh \ \beta = \frac{V}{kc \ \sqrt{1 - V^2/k^2 c^2}} \ .$$

For instance, eq. (208) describing the chance of R-circle into ellipse when going from rest reference frame to the moving one (IN VACUUM)

$$\frac{(x - 2R \ sh \ \beta)^2}{R^2 \ ch^2 \beta} + \frac{y^2}{R^2} = 1 \ , \tag{209}$$
$$ch \ \beta = \frac{1}{\sqrt{1 - V^2/c^2}}, \qquad sh \ \beta = \frac{V}{c \ \sqrt{1 - V^2/c^2}}$$

in case of the MEDIA is modified by presence of the parameter k:

$$\frac{(x - 2R \ sh \ \beta)^2}{R^2 \ ch^2 \beta} + \frac{y^2}{R^2} = 1 \ , \tag{210}$$
$$ch \ \beta = \frac{1}{\sqrt{1 - V^2/k^2 c^2}}, \qquad sh \ \beta = \frac{V}{kc \ \sqrt{1 - V^2/k^2 c^2}} \ .$$

20. Conclusions

Let us summarize some results.

I. *The influence of the relativistic motion of the reference frame on the light reflection law has been investigated in detail.*

The method used is based on applying the relativistic aberration affect for three light signals: incident, normal and reflected rays. Two choices for a normal light signal in the rest reference frame are used: one going to and another going from the reflecting surface. The form of the reflection law in the moving reference frame is substantially modified and includes an additional parameter which is the velocity vector of the reference frame. The relationship produced proves invariance under Lorentz group transformations. It is shown that the reflected ray, as measured by a moving observer, would not in general be in the same plane as the incident and normal rays. So a plane geometric figure, determined by the incident, normal and reflected rays, may be observed as having three dimensions.

II. *A general method to transform the form of any rigid surface in 3-dimensional space with respect to the arbitrary directed relative motion of the reference frame has been detailed.*

This method is based on the light signals processes and the invariance of the light velocity under Lorentz transformations. It is shown that a moving observer will measure a plane surface as a hyperboloid. That observer will also measure a spherical surface as an ellipsoid. A right line in the plane is seen by a moving observer as a hyperbola.

III. *Extending of the above analysis to the case of uniform media has been given.*

In the case of a uniform media the light velocity is not invariant under Lorentz transformation and is anisotropic. An alternative scheme of a modified Lorentz symmetry based on the invariance of the light velocity in the media is presented. This significantly modifies the formulation of the reflection law in the moving reference frame and also the form of any surface in the moving reference frame compared with the vacuum case.

IV. *Some geometrical aspects of the relativistic velocity concept in terms of the Lobachewsky 3-geometry are briefly discussed.*

References

[1] Einstein A. Zur Elektrodynamik der bewegten Korper. [The electrodynamic moving body] Annalen der Physik. 1905. Bd. 17. no 10. S. 891-921.

[2] Laub J. On the optics of moving bodies. Annalen der Physik. 1907. Bd. 23. no 9. S. 738-744.

[3] Laue M. The entrainment of light through moving bodies according to the relativity principle. Annalen der Physik. 1907. Bd. 23. no 10. S. 989-990.

[4] Strasser B. The Fizeau experiment on the alteration of the polarisation azimuth of a refracted ray by the earth's motion. Annalen der Physik. 1907. Bd. 24. no 11. 137-144

[5] Cantor M. The regulation of the light speed according to Fizeau and Acoustic analogues. Annalen der Physik. 1907. Bd. 24. no 13. S. 439-449.

[6] Laub J. The optical characteristics of moving bodies. Annalen der Physik. 1908. Bd. 25. no 1. S. 175-184.

[7] Sommerfeld A. The application of vector calculations on the basis of geometric optics. Annalen der Physik. 1911. Bd. 35. no 7. S. 277-298.

[8] Harnack A. The theory of movable mirrors. Annalen der Physik. 1912. Bd. 39. no 15. S. 1053-1058; Regarding the theory of movable mirrors. II. Annalen dder Physik.1914. Bd. 43. no 2. S. 295-308.

[9] Frank P. Use of the vector analysis on the geometrical optics in moving bodies. Annalen der Physik. 1917. Bd. 52. no 6. S. 649-656.

[10] Kennard E.H., Richmond D.E. On reflection from a moving mirror and the Michelson-Morley experiment. Phys. Rev. 1922. Vol. 19. P. 572-577.

[11] Guery F. Rev Gen. Electr. 1924. Vol. 16. P. 371-376.

[12] Titow A.M. The light reflectance of a moving sphere according to classical electrodynamics and the special theory of relativity. Zeitschrift fur Physik. 1925 . Bd. 33. S. 306-319.

[13] Eropkin D.I. Z. Phys. 1929. Bd. 58. n. 3-4. S. 268-272.

[14] Picht J. Z. Phys. 1926. Bd. 40. n.7. S. 521-529; 1930. Bd. 60. n.9-10. S.718.

[15] Majorana Q. Atti. Accad. Naz. Lincey Roma. 1934. Tom. 20. n.3. P. 754-759.

[16] H.E. Ives. The Doppler effect from moving mirrors. J.Opt.Soc.Am. 30, 255-257 (1940)

[17] Risco M. J. de Phys. ser.8. 1949. Vol. 10. n. 4. P. 128-131.

[18] Sivadjan J. Rev. Opt. (theor. et instr.) 1951. Vol. 30. n. 7. P. 323-331

[19] Schlomka T. Wiss. Z. Hochschule Archit. und Bauwesen Weimar. 1959-1960. Bd. 7. n. 2. S. 151-154.

[20] Nickle H.H. J. Appl. Phys. 1968. Vol. 39. n. 1. P. 353-355.

[21] Censor D. Scattering of a plane wave at a plane interface separating 2 moving media. Radio Science. 1969. Vol. 4. no 11. P. 1079.

[22] Bolotowskij B.M., Stoliarov C.N. Contemporary state of electrodynamics of moving medias (unlimited medias). Eistein collection, 1974. Moscow, Nauka. 1976. pages 179-275 (in Russian).

[23] V.Bolotovskii, S.N.Stolyarov. Wave reflection from a moving mirror and related problems. Usp. Fiz. Nauk. 1989, Vol. 159. n. 1. P. 155-180 (in Russian).

[24] He Q.C., Huang Y.X. Reflection of plane electromagnetic waves from the surface of a perfect conductor moving in an arbitrary direction. Chinese Science Bulletin. 2000. Vol. 45. no 17. P. 1564-1569.

[25] Puccini G.D., Selleri F. Doppler effect and aberration of light from the point of view of absolute motion. Nuovo Cimento. B. 2002. Vol. 117. no 3. P. 283-293.

[26] Gjurchinovski A. Aberration of light in a uniformly moving optical medium. Am. J. Phys. 2004. Vol. 72. no 7. P. 934-940.

[27] Gjurchinovski A. Reflection of light from a uniformly moving mirror. Am. J. Phys. 2004. Vol. 72. no 10. P. 1316-1324. Gjurchinovski A. Reflection of light from a uniformly moving mirror.

[28] Gjurchinovski A Einstein's mirror and Fermat's principle of least time. Am. J. Phys. 2004. Vol. Vol. 72. no 10. P. 1325-1327.

[29] Bernhard Rothenstein and Ioan Damian, Aberration associated with the reflection of light on a moving mirror. http://arxiv.org/abs/physics/0508084

[30] Landau L., Lifshitz E. The classical theory of fields. Addison-Wesley, Cambridge. 1951.

[31] Domenico Giulini. Algebraic and geometric structures of Special Relativity. arXiv:math-ph/0602018 v1 8 Feb 2006

[32] Ehrenfest P. The translation of deformed electrones and the surface theorem. Annalen der Physik. 1907. Bd. 23. no 6. S. 204-205.

[33] Einstein A. Comments on the note by Mr Paul Ehrenfest's "Translation of deformed electrones and the surface theorem". Annalen der Physik. 1907. Bd. 23. no 6. S. 206-208.

[34] Planck M. The dynamics of the moving system. Annalen der Physik. 1908. Bd. 26. no 6. S. 1-34.

[35] Max Born. Die Theorie des starren Elektrons in der Kinematik des Relativitätsprinzips. Annalen der Physik (Leipzig). 1909. Bd. 30. no 11. S. 1-56 [Born M. *The theory of the rigid electron in the cinematics of the relativity principle*.]

[36] Gustav Herglotz. Über den vom Standpunkt des Relativitätsprinzips aus als 'starr' zu bezeichnenden Körper. Annalen der Physik (Leipzig). 1910. Bd. 31. S. 393-415 [Herglotz G. From the view point of the relativity principle from bodies that have "star" distinctions.]

[37] Fritz Noether. Zur Kinematik des starren Körpers in der Relativitatstheorie. Annalen der Physik (Leipzig). 1910. Bd. 31. no. 5. S. 919-944. [On the understanding of the fixed bodies in the theory of relativity].

[38] Levi-Civita T. On Lorentz-Einstein's rigid fluctuations. Annalen der Physik. 1910. Bd. 32. no 6. S. 236-240.

[39] von Ignatowsky W. Fixed body and the relativity principle. Annalen der Physik. 1910. Bd. 33. no 13. S. 607-630.

[40] Born M. The definition of rigid bodies in the kinematics of relativity principles. Physikalische Zeitschrift. 1910. Bd. 11. S. 233-234

[41] Ehrenfest P. On the Mr. v Ignatowsky's treatment of Born's definition of inelasticity. Physikalische Zeitschrift. 1910. Bd. 11. S. 1127-1129.

[42] von Ignatowsky W. Remarks on the work "rigid bodies and the relativity principle". Annalen der Physik. 1911. Bd. 34. no2. S. 373-375.

[43] Hergoltz G. The mechanics of the deformable matter from the viewpoint of the relativity theory. Annalen der Physik. 1911. Bd. 36. S. 493-533.

[44] Max von Laue. Zur Diskussion über den starren Körper in der Relativitätstheorie. Physikalische Zeitschrift. 1911. Bd. 12. S. 85-87. [Laue M. On the discussion of rigid bodies in the theory of relativity.]

[45] Schaposohnikow K. On the relative dynamics of homogenous bodies. Annalen der Physik. 1913. Bd. 42. no 16. S. 1572-1574.

[46] von Laue M. On the dynamics of the theory of relativity, response to Mr Schaposchnikow. Annalen der Physik. 1913. Bd. 42. no 16. S. 1575-1579.

[47] Fokker A.D. On the space-time geometry of a moving rigid body. Reviews of Modern Physics. 1949. Vol. 21. P. 406-408.

[48] Gardner G.H.F. Rigid-body motions in special relativity. Nature. 1952. Vol. 170. P. 242-243

[49] Salzman G, Taub A.H. Born-type rigid motion in relativity. Phys. Rev. 1954. Vol. 95. P. 1659-1669.

[50] Newman E.T., Janis A.I. Rigid frames in relativity. Phys. Rev. 1959. Vol. 116. P. 1610-1614

[51] Ranesh Bhandhari. Visual appearance of a moving vertical line revisited. Am. J. Phys. 1961. Vol. 29. P. 67.

[52] Evett A.A. Relativistic rigid-body translations. Nuovi Cimento. 1963. Vol. 28. P. 685.

[53] Nevzglyadov V.G. Relativistic theory of motion in external field for a system with internal degrees of freedom. Nuovo Cimento. 1963. Vol. 29. P. 118.

[54] Boyer R.H. Rigid frames in general relativity. Proc. ROY. SOC. LONDON. A. 1965. Vol. 283. P. 343.

[55] WahlQuis H.D., Estabroo F.B. Rigid motions in Einstein spaces. J. Math. Phys. 1966. Vol. 7. P. 894.

[56] Williams G. Motions in relativistic spaces. J. Math. Anal. and Appl. 1968. Vol. 22. P. 646

[57] Mcgill N.C. Apparent shape of rapidly moving object in special relativity. Contemporary Physics. 1968. Vol. 9. no 1. P. 33.

[58] Ranesh Bhandari. Visual appearance of a moving vertical line. Am. J. Phys. 1970. Vol. 38. P. 1200.

[59] Bertsch G.F. Relation between classical and quantum mechanical rigid motion. Annals od Physics. 1971. Vol. 63. P. 534.

[60] Maugin G.Deformation fields of continuous medium in Minkowsi time-space interval. Comptes Rendus Hebdomadaires des Séances de l'académie des Sciences. 1971. Tome A 273. P. 65.

[61] Soderhol L. Relativistic motions isometric with respect to a particle. 1971 Physica Scripta. 1971. Vol. 4. P. 7.

[62] Gutzunay T.I. Problem of relativistically moving frames of references. Annales de l'Institut Henri Poincarée Sesction A. Physique théorique. 1972. Vol. 17. P. 71.

[63] Kafadar C.B. Dynamical problem for a rigid body in special relativity. Annalen der Physik. 1973. Bd. 29. S. 325-340.

[64] Newburgh R.G. Practical proposal for experimental study 0f Born rigid motions. Noovo Cimento. B. 1974. Vol. 23. P. 365-375.

[65] Creutz M. Quantum mechanics ofextended objects objects in relativistic field theory. Phys. Rev. D. 1975. Vol. 12. P. 3126-3144.

[66] Sorg M. Hyperbolic motion in finite-size model of radiating electron. Zeitschrift fur Naturforschung. SECTION A-A. 1975. Bd. 30. S. 1535-1542.

[67] Guzunajew Z.I., Kasatschkjw W.D., Terlezkij J.P. Use of relativistically moving systems in electrodynamics. Annalen der Physik. 1976. Bd. 33. S. 55-69.

[68] Pearle P. Absence of radiationless motions of relativistically rigid classical electron. Foundations of Physics. 1977. Vol. 7. P. 931-945.

[69] Barducci A., Lusanna L. Spherical rigid body - pole-dipole singularity. Nuovo Cimento. B. 1978. Vol. 47. P. 54-100.

[70] Barducci A., Lusanna L. Sperical rigid body, pole-dipole singularity. Nuovo Cimento. B. 1978. Vol. 47. no 1. P. 54-100.

[71] Kohler E. Schattner R. Some results on pseudo-rigid motions. General Relativity . and Gravitation. 1979. Vol. 10. no 8. P. 709-716.

[72] Sjodin T. Synchronization in special relativity and related theories. Nuovo Cimento. B. 1979. Vol. 51. no 2. P. 229-246.

[73] Kichenassamy S. The relativistically rigid motion of a surface. J. of Phys. A. Math. and Gen. 1982. Vol. 15. no 12. P. 3759-3762.

[74] Bona C. Rigid-motion conditions in special relativity. Phys. Rev. 1983. D. Vol. 27. no 6. P. 1243-1247.

[75] Bento L. Transverse-waves in a relativistic rigid body. International Journal of theoretical Physics. 1985. Vol. 24. no 6. P. 653-657.

[76] Peres A. Relativistic telemetry. 1987 Am. J. Phys. 1987. Vol. 55. no 6. P. 516-519; Peres A. Relativistic telemetry. Uspekhi Fuzicheskikh Nauk. 1988. Vol. 156. no 1. P. 145-151.

[77] Kevin G.Suffern. The apparent shape of a rapidly moving sphere. Am. J. Phys. 1988. Vol. 56. P. 729.

[78] Bel L., Martin J., Molina A. Rigid motion in special and general relativity. Journal of the Physical Society of Japan. 1994. Vol. 63. no 12. P. 4350-4363.

[79] Bel L., Llosa J. Spatially affine motions in relativity. Classical and quantum Gravity. 1995. Vol. 12. no 8. P. 1949-1967.

[80] Bel L., Llosa J. Meta-rigid motions and frames of reference. General Relativity and Gravitation. 1995. Vol. 27. no 10. P. 1089-1110.

[81] Barreda M; Olivert J Rigid motions relative to an observer: L-rigidity. International Journal of theoretical Physics. 1996. Vol. 35. no 7. P. 1511-1522.

[82] Dillon G. Electromagnetic mass and rigid motion. Nuovo Cimento. B. 1999. Vol. 14. no. 8. P. 957-972.

[83] Llosa J., Soler D. Reference frames and rigid motions in relativity. Classical and Quantum Gravity. 2004. Vol. 21. no 13. P. 3067-3094.

[84] Kim DH; Jo S.G. Rigidity in special relativity. J.of Phys. A. Math. and Gen. 2004. Vol. 37. no 15. P. 4369-4378.

[85] European Journal of Physics. 2004. Vol. 25. no 1. P. 123-126. Redzic D.V. Image of a moving sphere and the FitzGeraid-Lorentz contraction.

[86] Fedorov F.I. The Lorentz group. Moscow. 1979 (in Russian).

[87] Berezin A.B. Kurochkin Yu.A., Tolkachev E.A., Quaternions in relativistic physics. Minsk. 1989 (in Russian).

[88] A. Sommerfeld. Über die Zusammensetzung der Geschwindigkeiten in der Relativtheorie. Physikalische Zeitschrift. 1909. Bd. 10. S. 826-829.

[89] Alfred A. Robb. Optical Geometry of Motion: A New View of the Theory of Relativity. W. Heffer & Sons Ltd., Cambridge, 1911.

[90] Vladimir Varičak. Anwendung der Lobatschefskijschen Geometrie in der Relativtheorie. Physikalische Zeitschrift. 1910. Bd. 11. S. 93-96.

[91] Vladimir Varičak. *Über die nichteuklidische Interpretation der Relativtheorie.* Jahresberichte der Deutschen Mathematikervereinigung (Leipzig). 1912. Bd. 21. S. 103-127.

[92] Émile Borel. La théorie de la relativité et la cinématique. Comptes Rendus des séances de l'Académie des Sciences. 1913. Tom 156. P. 215-217.

[93] Abraham A Ungar. Beyond Einstein's Velocity Addition Law. volume 117 of Fundamental Theories of Physics. Kluwer Academic, Dordrecht, 2001; Abraham A Ungar. Analytical hyperbolic geometry. Mathematical Foundations and Applications. 2005. ISBN 981-256-457-8.

[94] Michels W.C., Patterson A.L. Phys. Rev. **60**, no 8, 589–591.

[95] Rosen H. Special theories of relativity. Amer. J. Phys. 1952. Vol. 20. P. 161-164.

[96] Intern. J. Theor. Phys. 1978. Vol. 17. no 3. P. 227-233. Stiegler K.D. Non-equivalency of right and left in electrodynamics of moving bodies - reply.

[97] V.M. Red'kov, George J. Spix. On the different forms of the Maxwell's electromagnetic equations in a uniform media. Proc. of XII Ann. Sem. Nonlinear Phenomena in Complex Systems. Minsk. 2005. P. 118-133; hep-th/0604080.